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QUESTIONS

& MENSURATION

TO SOLVE THEM.

BY THE

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PREFACE.

THOUGH this manual directly or indirectly embraces every branch of this science as generally found in text-books, its special object is to show how problems may be solved, and questions, asked in an inverse way, answered.

Arithmetic is a matter far more of common sense than generally supposed. Though rules are given in a concise form at the ends of the chapters when necessary, yet students are urged to study the science as little by rule as possible.

The question of units will be found more fully discussed than usual. To solve a problem by the adoption of some particular unit is a very different thing from solving one by what is called the unitary method, now so much used instead of the old method of proportion.

Though the unitary method is fully explained here, as required by the Education Code, still I believe that its supposed necessity is founded altogether on a mistaken notion.

The idea is that, in the old method of working a 'Rule of Three' question, it was necessary to multiply one concrete quantity by another. I maintain that this was no more done in the working of 'Rule of Three' than in so simple a question as: If 8 boys had 30 marbles each, how many marbles had they altogether?

Teachers' notice is particularly called to the method here given of finding the G. C. M. and L. C. M., and also of marking the point in multiplication, and especially in division, of decimals.

Other scales of notation than the denary are introduced, as I am convinced that nothing gives a grasp of the principles of value by position as their study.

The chapter, too, on Reciprocals will be found useful.

CONTENTS.

CATCH QUESTIONS IN ARITHMETIC AND HOW TO SOLVE THEM.

PART I.

CHAPTER I.

Notation — Numeration — Finding Remainders without Dividing—Notation and Numeration of Decimal Fractions—Other Scales than the Denary or Common—Complements—Signs and Methods of expressing Numbers.

1. If you happen to buy a railway ticket from a newly-opened series of a thousand, you will find the number printed thus, 005, or 016.[1] In writing down thousands (if there be any millions) and ones, write down the smaller numbers of thousands and ones thus, and you can never make a mistake in writing numbers: *e.g.* write down 4 millions 2 thousands and 6:—writing the 2 thousands, 002; and the 6, 006; the number becomes 4,002,006.[2]

2. Conversely, to read numbers, divide them off into periods of three (beginning from the right), and the millions, thousands, and ones read like £ *s.* and *d.*

3. Supposing we have a number, say 547, and we add on a 6 to it,—as we do in Long Division,—what operation or operations have we performed on it? We have multiplied it by ten and added 6 to the product. And why do we do this? To reduce it to the next denomination, just as we reduce the

[1] If the series were one of 10,000, there would be 4 figures.

[2] As a fact, railway tickets are not numbered from 1 to 1000, but from 000 to 999, so that 016 means the 17th ticket of the series. This enables 1000 tickets to be numbered with 3 figures, 10,000 with 4, and so on.

pounds to shillings, or tons to cwts., in the working of a question in Compound Long Division.

4. Next, supposing in 547 we insert a 2 between the 5 and 4 and a 3 between the 4 and the 7, thus, 52437: what have we done to it? (1) By inserting the 2 in 5247, we multiply the 5 hundreds by ten and add 2 hundreds to the result; and (2) by inserting the 3 we multiply the 524 tens by ten, and add on 3 tens more.

5. In the number 345719.

Since 10, and therefore 100, etc., are exactly divisible by 2 and 5, 34571 tens are exactly divisible by 2 and 5. If, then, we divide the *last figure* by 2 or 5, we have the remainder just as if we had divided the whole number by 2 or 5.

So, 345719 = some twos and 1;
or = some fives and 4.

Similarly, since 100, and therefore 1000, etc., are exactly divisible by 4, 10, and 25, the remainder, after dividing by these numbers, will be found by dividing the *last two figures* by 4, 10, or 25.

Thus, 345719 = some fours and 3;
or = some tens and 9;
or = some twenty-fives and 19.

Similarly, since 1000, and therefore 10,000, etc., are exactly divisible by 8, 100, 125, the remainder, after dividing by these numbers, will be found by dividing the *last three figures* by 8, 100, or 125.

Thus, 345719 = some eights and 7;
or = some hundreds and 19;
or = some hundred and twenty fives and 94.

6. If we write the number 345719 in this way—

$$\begin{aligned} 300000 &= 3 \times 99999 + 3, \\ 40000 &= 4 \times 9999 + 4, \\ 5000 &= 5 \times 999 + 5, \\ 700 &= 7 \times 99 + 7, \\ 10 &= 1 \times 9 + 1, \\ 9 &= \quad + 9, \end{aligned}$$

we see that

$$345719 = \text{some number of nines} + (3 + 4 + 5 + 7 + 1 + 9).$$

If, then, we wish to find the remainder after dividing by

3 or 9, all that we have to do is to find the remainder after dividing the sum of the digits[1] by 3 or 9.

Thus, 345719 = some nines and 2;
or = some threes and 2.

7. Again, if we write 345719 thus—

$$\begin{aligned} 300000 &= 3 \times 100001 - 3, \\ 40000 &= 4 \times 9999 \quad + 4, \\ 5000 &= 5 \times 1001 \quad - 5, \\ 700 &= 7 \times 99 \quad + 7, \\ 10 &= 1 \times 11 \quad - 1, \\ 9 &= \qquad\qquad + 9. \end{aligned}$$

Since 9999 (or any even number of nines) and 1001 (or any number of this form with an even number of ciphers between two ones) are exactly divisible by 11.

$$\begin{aligned} 345719 &= \text{some elevens} + \{(4+7+9) - (3+5+1)\} \\ &= \text{some elevens} + 0. \end{aligned}$$

Hence we obtain this useful rule to test the divisibility of a number by 11, or to find the remainder after dividing by 11.

From the sum of the digits in the odd places, subtract that of those in the even places, and divide the difference by 11.

If the sum of the digits in the even places be greater than that of those in the odd places, we must add on to the sum of the digits in the odd places as many elevens as are necessary to make it the greater of the two sums; *e.g.* in 90907081, we have to divide $(1+0+0+0)-(8+7+9+9)$ or $1-33$; we must therefore add on 33 to 1 to make it larger than 33, and we obtain the remainder 1. Thus $(1+33)-33=34-33=1$.

8. If there be no remainder after dividing a number by 2, and also none after dividing it by 3, there will be none after dividing it by 6. Now, to find the remainder after dividing a number by 6—

$$345719 = \text{some twos} + 1;$$

∴ the remainder, after dividing by 6, must be either 1 or 3 or 5.

Again, 345719 = some threes + 2;

∴ the remainder, after dividing by 6, must be either 2 or 5. Hence the remainder is 5.

9. If a number be exactly divisible by 3 and 4, it is exactly

[1] The digits of a number are the figures found in the number, irrespective of their position: thus, 4 and 5 are the digits of 45 and 54. Digit means finger.

divisible by 12, and the remainder, after dividing by 12, can be found in a similar manner to that of finding it after dividing by 6. Thus—

$$345719 = \text{some threes} + 2;$$

$\therefore$ the remainder, after dividing by 12, must be 2 or 5 or 8 or 11.

Again, $345719 = \text{some fours} + 3$;

$\therefore$ the remainder, after dividing by 12, must be 3 or 7 or 11, which shows us that $345719 = \text{some twelves} + 11$.

10. Since a number of this form, 357357 or $79079 = 1001 \times 357$ or 1001×79, and 1001 is exactly divisible by 7, 11, and 13, the number is exactly divisible by 7, 11, or 13.

11. Testing the divisibility of numbers (except multiples of 1001) by 7 is generally considered impracticable. The following method is added as an intellectual and interesting exercise. First divide 1 followed by several ciphers by 7, and put down the remainders underneath the corresponding figures of the quotient, thus—

$$\begin{array}{l} 7\underline{)100000000,\ \text{etc.}} \\ 014285714,\ \text{etc.} \\ 132645132,\ \text{etc.} \end{array}$$

These remainders, 1, 3, 2, etc., show us the remainders after dividing 1, 10, 100, etc., by 7;

$\therefore$ 345719 may be written thus—

$$\begin{array}{rlll} 9 = & & & \text{one seven} + 2, \\ 10 = & 1 \times (\text{one seven} + 3) = & \text{some sevens} + 3 = & \text{some sevens}, \\ & & & + 3, \\ 700 = & 7 \times (\text{some sevens} + 2) = & \text{,,} \quad + 14 = & \text{,,} \quad + 0, \\ 5000 = & 5 \times (\quad \text{,,} \quad + 6) = & \text{,,} \quad + 30 = & \text{,,} \quad + 2, \\ 40000 = & 4 \times (\quad \text{,,} \quad + 4) = & \text{,,} \quad + 16 = & \text{,,} \quad + 2, \\ 300000 = & 3 \times (\quad \text{,,} \quad + 5) = & \text{,,} \quad + 15 = & \text{,,} \quad + 1. \end{array}$$

The remainder, therefore, will be found by dividing

$$2+3+0+2+2+1 \text{ or } 10 \text{ by } 7,$$

which gives a remainder 3. Hence the rule—

Multiply the units	by 1,	and cast out the sevens, if any.
,, tens	,, 3	,,
,, hundreds	,, 2	,,
,, thousands	,, 6	,,
etc.		etc.

To take another example.

To find the remainder after dividing 372869541 by 7. Placing the multipliers underneath, and writing down the remainder only, we have

$$\begin{array}{c} 372869541 \\ 231546231, \end{array}$$

giving $1+5+3+5+3+5+2+0+6=30$;

$\therefore$ remainder $=2$.

12. A similar method could be used for any number, as 13 or 17, 19 or 23, etc., the multipliers being found by dividing 100 .... by 13, or 17, or 19, and putting down the remainders in their order underneath the corresponding figures of the quotient. As we can have remainders up to 12, 16, and 18 respectively, we must leave room and write commas between the figures, thus—

$$\begin{array}{l} 13\overline{)1\ \ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 0\qquad} \\ \quad 0\ \ 0\ 7\ \ 6\ 9\ 2\ 3\ \ 0\ 7, \text{etc.} \\ \quad 1, 10, 9, 12, 3, 4, 1, 10, 9, \text{etc.} \end{array}$$

Find, without division, the remainder after dividing 47689123 by 13.

$$\begin{array}{c} 4\quad 7\quad 6\quad 8\quad 9\quad 1\quad 2\quad 3 \\ 10\,.\,1\,.\,4\,.\,3\,.\,12\,.\,9\,.\,10\,.\,1 \\ 1+7+11+11+4+9+7+3=53; \end{array}$$

therefore the remainder will be 1.

13. Decimal Fractions are said to have been discovered in this way. A teacher wrote on the board a series of numbers, and made his pupils calculate the positional value of any figure by its connection with figures that were written before it. Thus, if in 347218 I know in any way the 7 to be thousands, I therefore know the 2 to be hundreds, because the figure is placed next to the right of the thousands. I know the 8 to be units, because it is the third figure from the 7, or the thousandth of a thousand. Hence, if I know a figure to be units, the next figure to the right must be tenths, and the next hundredths, etc. All that is wanted, then, is some mark to show where the units are. The point, then, that we use (generally called the decimal point) is really the unit point, and shows us where the units are, viz. the figure to its left.

14. A decimal may be read, irrespective of the point, and

called by the name of its right-hand figure. Thus, 135·4706 is read 1 million 354 thousand 706 tenths of thousandths.

15. Conversely, a decimal may be written down as a whole number, and the last figure made, by its position with regard to the point, whatever the denomination of the number may be. Thus, to write 1 million 4 thousands and 20 hundredths of thousandths, all we have to do is to write down 1004020; and since the right-hand figure is hundredths of thousandths, it must be the fifth figure from the point, and the number is thus expressed, 10·0402, the final cipher being valueless.

16. Though 347654 usually represents 4 ones 5 tens 6 hundreds, or ten times ten, 7 thousands or ten hundreds or ten times ten times ten, and so on, there is no reason why it should not have represented 4 ones, 5 eights (or any other number larger than 7, the largest digit), 6 eight times eights, 7 eight times eight times eights, and so on.

17. It is an exceedingly useful exercise to work numbers in all the scales. In this chapter we shall only learn to reduce numbers from or to the ordinary (10) scale to or from that of some other number or radix.

18. Let 347654 be any number expressed in the ordinary (10) scale, and let it be required to reduce this to another—say octonary (8)—scale. We will write eight times eight or sixty-four thus—eight^2; and eight times eight times eight, or five hundred and twelve thus—eight^3, and so on.

8)347654
8)43456 eights, 6 ones.
8)5432 eight^{2}s + 0 eights.
8)679 eight^{3}s + 0 eight^{2}s.
8)87 eight^{4}s + 3 eight^{3}s.
8)10 eight^{5}s + 7 eight^{4}s.
1 eight^6 + 2 eight^{5}s.

Hence the number becomes

1273006.

To change this number back into its equivalent expressed

in common (10) scale. Multiply it by a series of eights, adding in the next denomination, just as you reduce £ *s. d.* to farthings, or tons, cwts., quarters, etc., to ounces, each denomination being 8 times as large as the next.

1, 2, 7, 3, 0, 0, 6		1, 2, 7, 3, 0, 0, 6
8		8
10 eight^{5}s	or omitting the	10
8	multiplier after	87
87 eight^{4}s	the first.	679
8		5432
679 eight^{3}s		43456
8		347654
5432 eight^{2}s		
8		
43456 eights		
8		
347654 ones.		

19. 37·256 in the octonary scale means 3 eights, 7 ones, 2 eighths, 5 sixty-fourths, and 6 five hundred and twelfths.

To reduce 134·453125 from denary (10) to octonary. First reduce the 134 as in 18.

$$
\begin{array}{r}
8\underline{)134} \\
8\underline{)16\cdot 6} \\
2\cdot 0
\end{array}
$$

∴ 134 denary = 206 octonary.

To reduce the decimal fraction ·453125 denary to its equivalent in the octonary. Since, then, 8 eighths is a unit and eight sixty-fourths an eighth, etc., the following is evident.

·453125 units
8

3·625000 eighths or 3 eighths + $\frac{625}{1000}$ of another eighth.
8

5·000 sixty-fourths or 5 sixty-fourths;

∴ ·453125 denary = ·35 octonary.

Therefore to reduce ·35 octonary to the denary, we must reverse this operation.

$$\begin{array}{l} 8\underline{)5\cdot00} \text{ sixty-fourths} \\ 8\underline{)3\cdot625} \text{ eighths} \\ \quad \cdot453125 \text{ units,} \end{array}$$

or ·35 octonary = 453125 denary.

To reduce, then, a number from one scale, say septenary (7) to another, say quinary (5), reduce the number expressed in septenary to that in denary, and then reduce it from the denary to the quinary; *e.g.* reduce 135·46 septenary to the quinary.

To deal with the whole number first.

$$\begin{array}{ll} 135 & \\ \underline{7} & \\ 10 \text{ sevens} & \\ \underline{7} & \\ 75 \text{ ones} & \end{array} \qquad \begin{array}{l} 5\underline{)75} \\ 5\underline{)15} \text{ fives} + 0 \\ 3 \text{ (five)}^2\text{s} + 0 \end{array}$$

∴ 135 septenary = 300 quinary.

Now to deal with the fractional part, ·46.

$$\begin{array}{l} 7\underline{)6\cdot} \text{ forty-ninths} \\ 7\underline{)4\cdot857142}, \text{ etc., sevenths} \\ \quad \cdot693877, \text{ etc., units.} \end{array}$$

And to reduce this to the quinary.

$$\begin{array}{rl} \cdot693877 & \text{units} \\ \underline{5} & \\ \underline{3\cdot469385} & \text{fifths} \\ \underline{5} & \\ 2\cdot34692 & \text{twenty-fifths} \\ \underline{5} & \\ 1\cdot7346, & \text{etc., fifth}^3\text{s;} \end{array}$$

∴ ·46 septenary = ·321, etc., quinary.

20. By means of the usual signs +, −, etc., we can write down the same number in several different ways.

Thus, $9 = (1)\ 7+2,\ (2)\ 10-1,\ (3)\ 3\times3,\ (4)\ 18\div2,\ (5)$ £27 : £3, (6) $\frac{36}{4}$, (7) $\sqrt[2]{81}$, etc.

We can frequently shorten our operations by writing down numbers in one of these forms.

21. The complement of a number with respect to another number is what it wants to complete that number. Thus—with respect to 10—7, 8, 4 are the complements of 3, 2, and 6. With respect to 12—4, 5, 9 are the complements of 8, 7, and 3 respectively. The quickest way to add numbers is to split them up, so as to stop at all the tens.

$$\text{Thus, } 8+4+7+6+9+7, \text{ etc.,}$$
$$\text{or, } 8+2+2+7+1+5+5+4+6+1=41.$$

When we split up in thought the 4 into 2 (complement of 8) and 2, the 6 into 1 (complement of 9) and 5, etc.

Examination and Examples.

1. What is the principle by which the ten symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 are used to represent any number, however large or small?

2. Why may you add on ciphers to the left of a whole number and to the right of a decimal, without altering its value?

3. What effect has it to add a number as 7 on to a number as 453?

4. In 4572 insert a 3 between the 4 and the 5, and state what alteration you have effected on the number.

5. Show that the number 360360 will divide exactly by all the numbers from 2 to 15 inclusive.

6. Find (without dividing) the remainders after 342213 has been divided by 9, 11, and 8.

7. Find (without dividing) the remainders after 253417 has been divided by 6 and 12.

8. Find (without dividing) the remainder after 49372 has been divided by 7.

9. Write down the number 11 in 6 different ways; only using the same symbol once.

10. State clearly what you understand by the number 347 expressed in the ordinary scale.

11. What would 347 mean in the octonary (8) scale?

12. Reduce 347 octonary to denary.
13. Reduce answer of 12 into the quinary (5).
14. Reduce 41312 (quinary) into the denary.
15. Reduce answer of 14 into the nonary (9).
16. Reduce 36134 septenary into the denary.
17. Reduce answer of 16 into the quaternary (4).
18. Reduce 5718 duodenary into the denary.
19. Reduce 216 septenary to denary.
20. Reduce answer of 19 to binary (2).
21. Reduce 101101 (binary) to denary.
22. Reduce answer of 21 to duodenary (12).
23. State clearly what you mean by ·475 (common scale).
24. How were decimals discovered?
25. Write down in words 3·12.
26. Write down in words 47·681.
27. Write down in words ·003754.
28. Write down in words 100·1.
29. Write down decimally, 1 thousand and 20 millionths.
30. " " " 14 thousand and 37 thousandths.
31. " " " 29 thousandths and 219 tenths of thousandths.
32. Write down decimally, 1 million 4 thousand and 27 hundredths of thousandths.
33. Explain clearly what is meant by the fraction ·346 in the septenary scale.
34. Reduce ·36 denary to quinary.
35. Reduce ·244 quinary to denary.
36. Reduce ·75 denary to quaternary.
37. Reduce ·323 quaternary to denary.
38. Reduce 275·625 denary to octonary.
39. Reduce 34·133 quinary to denary.
40. Reduce 243·376 denary to quinary.
41. Divide 10000000 by 17, inserting the remainders under the quotient. Hence find remainder (without dividing) after dividing 47326121 by 17.
42. Write down the complements with respect to 10, of 6, 9, 4.
43. Write down the complements with respect to 12, of 7, 4, 8.
44. Write down the complements with respect to 100, of 43, 2, 77.

45. What are the advantages and disadvantages of the denary scale?

46. Show that if we reverse any two figures, the difference between the two numbers will always divide by 9; *e.g.* $74-47=27$.

47. Show that if the digits of any number be written in any order, the difference between this and the original number will divide by 9.

48. Connect the term digit with the ordinary scale.

49. In the number 340162 septenary, where must I place the unit point that the 4 may represent 4 forty-nines?

50. Find the remainder (without dividing) after the number 327131 has been divided by all numbers from 2 to 15 inclusive.

APPENDIX TO CHAPTER I.

A number is divisible without remainder by

2, 5, 10	if the last figure is exactly divisible by	2, 5, 10
4, 25	if the last 2 figures are exactly divisible by	4 or 25
8	„ 3 figures „	8
3, 9	if the sum of the digits is exactly divisible by	3, 9
6	if the number is exactly divisible by 2 and 3	
12	„ „ 4 „ 3	

11 if the sum of the digits in the odd places (the units, hundreds, etc.) be equal to or greater by a multiple of 11 than the sum of the digits in the even places.

7 if the digits in the units, tens, etc., be respectively multiplied by 1, 3, 2, 6, 4, 5, 1, 3, 2, etc., and their sum is divisible by 7. This test is only introduced as a mental exercise and a matter of interest, not being practically useful.

CHAPTER II.

Four Common Rules in Integers, Decimals, and other Scales than the Denary—Proofs of Rules—Short Methods.

1. A question such as 3 times 7 may be asked in a reverse way: thus, What times 7 make 21? or, what times what make 21? When a question is asked in this way, it is best to replace the numbers by very small ones and analyse the process by which your mind arrives at the answer.

Supposing I were asked what number must be added to 375 so that 14 times the sum is greater by 7 than 29 times 245. Let us ask the same question with very small numbers.

What number must be added to 1 so that twice the result is greater by 6 than 3 times 4? The mind at once realizes that it is 8. First, it sees that 18 is 6 greater than 3×4; and secondly, that 18 is 2×9; and thirdly, that 8 must be added to 1 to make it 9.

So what we have really done is seen in the expression,

$$[\{(3 \times 4) + 6\} \div 2] - 1 = 8.$$

Hence the following similar arithmetical statement will show what we are to do with the larger numbers, which the mind cannot grasp so easily.

$$[\{(29 \times 245) + 7\} \div 14] - 375 = 133,$$

the 133 being thus found.

$$\begin{array}{rr}
 & 245 \\
 & 29 \\
\hline
 & 2205 \\
 & 490 \\
\hline
245 \times 29 = & 7105 \\
\text{add} & 7 \\
\hline
14) & 7112(508 \\
 & 70 \\
\hline
 & 112 \\
 & 112 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 508 \\
\text{less} & 375 \\
\hline
\text{leaves} & 133 \\
\hline
\end{array}$$

Now to prove this—

$$\begin{array}{r}
375 \\
133 \\
\hline
508 \\
14 \\
\hline
2032 \\
508 \\
\hline
7112 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{lr}
 & 7112 \\
\text{less} & 7 \\
\hline
 & 7105 \\
\hline
\end{array}$$

and $7105 = 29 \times 245$

2. Questions are asked in this way sometimes:—What is the least number of 6 digits that will exactly divide by a number, say 347? The least number that can be expressed by 6 digits is 100000; but since this will not divide exactly by 347, let us add 347 on to it and then subtract the remainder (which must be less than 347), and this will give us the number required. Thus:—

$$\begin{array}{r}
347)100347(289 \\
694 \\
\hline
3094 \\
2776 \\
\hline
3187 \\
3123 \\
\hline
64;
\end{array}$$

∴ the number required is 100283.

Similarly the largest number of 6 digits can be found by dividing 1000000 and subtracting the remainder (which must be less than 347) from the dividend.

3. If we are asked for the number nearest (greater or less, as the case may be) to another number that will exactly divide by a given divisor; it is better to divide the number as it stands, and if the remainder be less than half the divisor, subtract it from the number; but if greater, add the complement of the remainder, with respect to the divisor, to the dividend; this will make the quotient greater by 1. Thus—

Find the nearest number to 19575 and 19825 which will exactly divide by 391.

$$391)19575(50$$
$$\underline{1955}$$
$$25$$

Since 25 is less than half of 391 we must take away 25 from 19575, which gives 19550, the number required.

$$391)19825(50$$
$$\underline{1955}$$
$$275$$

Since 275 is greater than half of 391 we must add 116 (391–275), which will give 19941, the number required.

4. To find the remainder after dividing by means of factors. The method of changing numbers from the denary scale into any other will suggest the proof.

Let it be required to divide 472613 by 7, 9, 11, or 693.

7)472613 ones.

9)67516 sevens and 1 one.

11)7501 sixty-threes and 7 sevens.

681 six hundred and ninety threes and 10 sixty-threes.

Ans. 681 and 1 + 49 + 630 or 680.

5. The remainder after dividing must be of the same denomination as the dividend: so that if one were asked what the remainder would be after dividing 1·0001 inches by any number, the answer is so many tenths of thousandths of an inch. The quotient is found by reading the divisor with the denomination of the dividend; thus supposing we were dividing square yards by $30\frac{1}{4}$ or $\frac{121}{4}$ to reduce them to square poles. First, we multiply the number of yards by 4 to reduce them to quarter yards, and then we first divide these quarters by 11, getting so many 11 quarter yards, and secondly by another 11 to reduce these 11 quarter yards to 121 quarter yards or poles.

6. The operation of dividing by $5\frac{1}{2}$ can be done in one effort. If a remainder, as we perform the operation, contain a $\frac{1}{2}$, it must of course be carried as a 5 to the next denomination. If there be a remainder of 5, and the next digit contain a 5, this 5 must be added on to the former digit in the form of a half to make the number divisible by $5\frac{1}{2}$, or the next quotient will be 10. The working is thus, the remainders being inserted under the corresponding figures in the quotient,

though, of course, this need not be done in the ordinary course—

$$
\begin{array}{rcccccccccc}
5\tfrac{1}{2}) & \underline{3} & \underline{6} & \underline{9} & \underline{2} & \underline{6} & \underline{9} & \underline{4} & \underline{9} & \underline{4} \\
 & & 6 & 7 & 1 & 3 & 9 & \underline{9} & 0 & 8 \\
 & & 3, & \tfrac{1}{2}, & 1\tfrac{1}{2}, & 4\tfrac{1}{2}, & 4\tfrac{1}{2}, & \tfrac{\bar{1}}{2}, & 4, & 0
\end{array}
$$

When we divided the thousands we had to divide $5\frac{1}{2}$ into 54, which gave a quotient 9 and $4\frac{1}{2}$ over; this is added on to the 4 as 45, making 49, which gave a quotient 8 and 5 over; but seeing that the next digit contains a 5, I added 5 on to the 49 in the form of $\frac{1}{2}$; so that I divided $49\frac{1}{2}$, which gave the quotient 9 with a remainder $\frac{\bar{1}}{2}$. By $\frac{\bar{1}}{2}$ we mean that $\frac{1}{2}$ of one of the higher denomination has been taken from the next figure in the form of 5, hence instead of adding on 5 we subtract 5 from 9 and get 4, into which $5\frac{1}{2}$ will not divide.

This is a useful exercise, and makes students realize the meaning of those operations they perform almost mechanically.

7. By examining the digits of a multiplier we can often reduce the number of lines necessary. *e.g.* Supposing I want to multiply by 147, since 14 is 2×7, if after multiplying the multiplicand by 7 I double this line and place the result underneath it in its proper place, I reduce the operation from three lines to two. Thus—

$$
\begin{array}{rll}
 & 371 \times 147 & \\
 & \underline{7} & \\
 & 2597 & = 7 \text{ times} \\
\text{1st line} \times 2 = & \underline{5194} & = 140 \text{ times} \\
 & \underline{\underline{54537}} & = 147 \text{ times}
\end{array}
$$

To take a longer example—

Multiply 47162 by 729819 in three lines, the working will be thus—

$$
\begin{array}{rll}
 & 47162 & \\
 & \underline{9} & \\
 & 424458 & \text{or } 9 \text{ times} \\
\text{1st line} \times 9 = & 3820122 & \text{or } 810 \text{ times} \\
\text{2nd line} \times 9 = & \underline{34381098} & \text{or } 729000 \text{ times} \\
 & 34419723678 & = 729819 \text{ times,}
\end{array}
$$

where the second line is got from the first by multiplying it by 9, and the third from the second in the same way. In 1448412, for first line we should multiply by 12, for second multiply the first line by 7, and for third line multiply the first line by 12, taking care the second line is two places and the third four places respectively to the left of the units.

8. Since 9 can be written down in the form of $10-1$.

To multiply a number by 9, add on a cipher and subtract the number from it. The number need not be re-written. Understand a cipher at either end, and subtract each figure from the one that succeeds it, beginning by subtracting the units from an added 10.

Thus: 347×9, 7 from 10 leaves 3; 5 from 7, 2; 3 from 4, 1; and 0 from 3, 3; $\therefore$ the product is 3123.

Similarly 11 may be written $10+1$; hence add on a cipher at either end and add every figure to the one that succeeds it, thus—

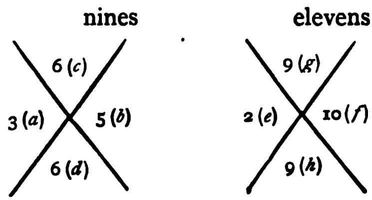
$$347 \times 11,\ 7+0=7,\ 4+7=11, \text{ carry the 1 to } 3+4=8,\ 0+3=3;$$

$$\therefore \text{ the product is } 3817;$$

again $25=\frac{100}{4}$; $\therefore$ to multiply by 25 add on two ciphers and divide by 4; and to divide, multiply by 4 and cut off two figures; the two figures cut off will be so many hundredths of a unit.

9. To prove a result in Multiplication by casting out nines or elevens, or in fact any number. Find the remainder after the multiplicand, multiplier, and product have been respectively divided by 9 or 11, or the number chosen; then multiply the remainders of the multiplier and multiplicand, and see what the remainder of their product would be; and if this corresponds with that already found for the product, the product is right or wrong by 9 or 11 or the number used. If it is right or wrong both by 9 and by 11, it must he right or wrong by 99. To take an example—

$$\begin{array}{r} 453 \\ 329 \\ \hline 4077 \\ 906 \\ 1359 \\ \hline 149037 \end{array}$$



The small letters show whence the numbers come.

Now 453 = some nines + 3 (*a*), and 329 = some nines + 5 (*b*), and 149037 = some nines + 6 (*c*). But 3×5 = some nines and 6 (*d*), therefore we are right or wrong by some number of nines.

Again, 453 = some elevens + 2 (*e*), and 329 = some elevens + 10 (*f*), and 149037 = some elevens + 9 (*g*) and 2×10 = some elevens + 9 (*h*), hence the product is correct or wrong by some number of elevens, that is, it is right or wrong by 9×11, or 99.

Since the remainder, after dividing by such numbers as 2, 4, 5 is found by examining the later figures only, this method of proving the correctness of a product would not affect the higher denominations at all.

This is the proof of the test—

$$\begin{array}{r l}
453 & = \text{some nines} + 3 \\
\underline{329} & = \underline{\text{some nines} + 5} \\
\underline{149037} & \quad \text{some nine}^2\text{s} + 3 \text{ times some nines} \\
 & \qquad\qquad\quad + 5 \text{ times some nines} + 15 \\
 & \overline{\quad \text{some nine}^2\text{s} + \qquad\quad \text{some nines} + 15} \\
 & \overline{\quad \text{and } 15 = \text{a nine} + 6\,;} \\
\multicolumn{2}{r}{\text{but } 149037 = \text{some nines} + 6.}
\end{array}$$

10. Before working questions in scales, it is as well to write out the Multiplication Table in the notation of the scale, *e.g.* to write out the Multiplication Table in the septenary scale—

1	2	3	4	5	6	7 or 1·0
2	4	6	1·1	1·3	1·5	2·0
3	6	1·2	1·5	2·1	2·4	3·0
4	1·1	1·5	2·2	2·6	3·3	4·0
5	1·3	2·1	2·6	3·4	4·2	5·0
6	1·5	2·4	3·3	4·2	5·1	6·0
7 or 1·0	2·0	3·0	4·0	5·0	6·0	1·0·0
8 „ 1·1	2·2	3·3	4·4	5·5	6·6	1·1·0
9 „ 1·2	2·4	3·6	5·1	6·3	1·0·5	1·2·0
10 „ 1·3	2·6	4·2	5·5	7·1	1·1·4	1·3·0
11 „ 1·4	3·1	4·5	6·2	7·6	1·2·3	1·4·0
12 „ 1·5	3·3	5·1	6·6	8·4	1·3·2	1·5·0

Let us multiply 13021 by 4653, both expressed in septenary scale. The operation will be perfectly simple, thus—

$$\begin{array}{r} 13021 \\ 4653 \\ \hline 42^{a}0\ 63 \\ 10\ 11\ 3^{b}5 \\ 114^{c}15\ 6 \\ 551\ 14 \\ \hline 1006\ 16\ 3\ 4^{d}3 \\ \hline \end{array}$$

(*a*) Here we got 9, which is 1 seven and 2 ones of the denomination, viz. 7^3. (*b*) Here we got 10, which is 1 seven and 3 ones of the denomination, viz. 7^2. (*c*) Here we have multiplied 3 by 6, which the table shows us is 2·4, hence we write down the 4 and carry the 2. (*d*) Here $5+6=11$, which contains 1 seven (seven^2) and 4 (7)s; so we write down the 4 and carry the 1 to the $(7)^3$s. To prove this, reduce the quantities

to the denary, multiply and reduce the product back to the septenary, and compare thus—

```
 13021 (sept.)        4653 (sept.)        3445 (denary)
 7                    7                   1704 (denary)
 10                   34                 13780
 70                  243                24115
 492                1704 (denary)       3445
3445 (denary)                          5870280 (denary).

7)5870280 (denary)
 7)838611   3
 7)119801   4
  7)17114   3
   7)2444   6
    7)349   1
     7)49   6
      7)7   0
        1   0
```

Since, then, 5870280 denary = 100616343 septenary, we know our working is right.

11. If we look at the above Multiplication Table we find that the units in the 6 column decrease by 1, and that in the other figure they increase by 1, just as the 9 column do in the ordinary Multiplication Table, and we shall show that the properties of 9 in the denary scale are common, in all the scales, to the number next below the radix. Similarly, too, we shall show that in septenary or any scale, what is proved of 11 with regard to the denary is true of 8 (1·1) with regard to septenary, or of 9 with regard to octonary, etc.

12. In doing a Long Division question one can sometimes

save time by writing down the Multiplication Table of the divisor, thus—

```
387)45612371(117861
     387
     ---
      691
      387
      ----
      3042
      2709
      -----
       3333
       3096
       -----
        2377
        2322
        -----
          551
          387
          ---
          164
```

Numbers	
1	387
2	774
3	1161
4	1548
5	1935
6	2322
7	2709
8	3096
9	3483

The numbers 774, 1161, etc., are written down very rapidly, as they can be got by very simple operations on those found before; *e.g.* the 4 line is 2nd × 2, the 7 line is 4th + 3rd, the 9 3rd × 3, etc. The advantage of this method is that it prevents the possibility of getting a remainder greater than the divisor, which would require correction. It is particularly useful in working in other scales than denary.

13. Since decimals are only fractions expressed as whole numbers, they can be worked as whole numbers, irrespective of the unit or decimal point; and a moment's reflection will tell us at once where the units in the results are, *e.g.*—

To add 4·45, 17·021, ·0067. In the third quantity there are no units expressed. We think it is far better to insert them thus, 0·0067. Now use the rule for whole numbers, viz. place units under units, etc., and we get at once—

4·45
17·021
0·0067
———
21·4777
———

Since 4 units + 7 units gives us 11 units, we know that the 1 must be units in the sum, and so place the unit point after it to fix it as a unit.

Similarly Subtraction.

From 3·1476 take ·49219, insert the units in subtrahend and place them as in whole numbers, adding on or understanding a cipher after the 6 in the minuend.

3·1476
0·49219
———
2·65541
———

Having had to add on a unit in the form of ten-tenths to make the tenths possible to be subtracted, I must add it on to the cipher before I subtract the cipher from the 3, which must leave two units; hence the position of the unit point.

14. In Multiplication, since units × units must be units, that is, 3 ones taken 5 ones of a time = 15 ones. Supposing that the figures are properly arranged, I have only to notice the line where the number is which is produced by multiplying the units by units, and it must be the unit line.

Thus, to multiply 13·7 by 1·006—

13·7
1·006
———
822
1·37
———
13·7822
———

Now, since the line which contains the product of 3 units by 1 unit is the fifth from the right, the 3 in the product must be units, hence the position of the unit point. To take another example—·006 × ·003, insert the units thus, when it is seen directly that the cipher obtained by multiplying the unit by the unit is the seventh from the right, hence the position of the point.

0·006
0·003
———
18
0·00000
———
0·000018
———

15. Division, too, of decimals can be reduced to a very simple matter of common sense. Since 1 unit is contained in 6 units 6 units of a time, and any number, say 2 thousandths,

is contained in a number of like denomination, say 8, 4 units of a time. We see at once that the quotient must be unity when we are dividing any denomination by a number of like denomination.

Here are three examples generally requiring three rules. (1) 47·125 ÷ 2·5; (2) 47·125 ÷ ·0000025; and (3) 47·125 ÷ 250000.

```
(1) 2·5′)47·1′25(18·85
        25
        ——
        221
        200
        ———
         212
         200
         ———
          125
          125
          ———
```

I am dividing by tenths, hence I mark the tenth in the dividend, and when I divide this tenth I obtain 8 units, which I immediately mark with the unit point.

(2) ·0000025′)47·1250000′(18850000.

In this case I am dividing by tenths of millionths, so I add on ciphers that I may have a tenth of a millionth in the dividend, which I mark; and when I divide this figure I obtain units, which I mark. (I could leave it unmarked; as, being the last figure of an integer, it must be a unit.)

(3) 250000′)47′·1250000(0·0001885.

As I am dividing by units I mark the 7, which gives me no units, which I write down and mark with the unit point; but 471 will not divide by 25000, hence I must insert another cipher, and so on until the fourth figure of the decimal.

16. All that is said in the foregoing examples about decimals is true of fractions expressed in other scales.

17. The difference of two quantities is unaltered if the same quantity be added to both. Thus the difference between 7 and 2 is 5, but the difference between 7 + 4 (or 11) and 2 + 4 (or 6) is also 5; hence, if I am subtracting one quantity from another and I have to subtract a larger quantity from a less, I

add one of the next denomination to both the minuend and subtrahend, adding it to the former in the form of the denomination in which the number is being subtracted from the other, thus—

£	s.	d.
3	2	4
1	17	8
1	4	8

In which I add on a shilling in the form of 12 pence to the minuend and in the form of 1 shilling to the subtrahend, and again I add on £1 in the form of 20 shillings to the minuend, and in that of £1 to the subtrahend.

18. If a quantity is to be divided amongst several people where some are to have more than others, it is necessary to take away all these extra portions from the quantity before the division takes place, thus—

Divide £100 amongst A, B, C, D, so that C has £2 more than D, B has £3 more than C, and A £4 more than B. Since C has £2 more than D, and B has £3 more than C, or £5 more than D, and A has £4 more than B, or £9 more than D, we must take these £16 away and divide the £84 left equally amongst the four, and then add on the £2, £5, £9 afterwards; so that D has £21, C £23, B £26, A £30.

Examination and Examples.

1. What number subtracted from 347612 leaves a remainder 39875?

2. Explain clearly how you get over the difficulty of subtracting the larger number 5 from the smaller number 2 in the first question.

3. What number must be added on to the sum of 4761 and 398 so as to give the entire sum of 100000?

4. The subtrahend of a question is 47612 and the difference 59163. What is the minuend?

5. What number divided by 379 gives a quotient 9734 and a remainder 57?

6. What number must be added to 1000000 to make it exactly divisible by 492?

7. Find the nearest number to 57432 which will divide exactly by 379.

8. Find the nearest number to 10000 which will divide by 791 with a remainder 472.

9. Divide (Short Division) 347121 by $5 \times 7 \times 9$, and explain clearly how you obtain the remainder.

10. Find the remainder in question 9 without dividing.

11. Explain the method of proof of Multiplication by casting out nines. What mistakes does it fail to detect?

12. Why would proving a Multiplication question by casting out eights be of little value as a proof?

13. Write out the Multiplication Table up to 12 times 7 in the octonary scale, and multiply 342615, octonary, by 423 (octonary) and also by 12 (denary) in one line.

14. Prove (without division) that 43362 (septenary) is divisible by 6.

15. Prove (without division) that 45342 (nonary) is exactly divisible by 10 (denary).

16. Prove (without division) that 357357 (nonary) is exactly divisible by 5 and 10 (denary).

17. Multiply (only using 3 lines) 47169 by 172814412.

18. Multiply (only using 3 lines) 47169 by 152412.

19. Multiply (only using 3 lines) 47169 by 1332221.

20. What number must be added to 479 so that 75 times the sum may be less by 30 than 95 times 399?

21. Express 9 in four different ways, only using a symbol once; and multiply 47 by it in all the ways.

22. How much does £45, 6s. $7\frac{1}{2}$d. want of £50?

23. Find the complement with respect to £10 of £8, 6s. $7\frac{1}{2}$d.

24. What is the difference between losing and winning a bet of £5, 7s. 7d.?

25. What is the difference between being 5 minutes too soon and 10 minutes too late?

26. If I start on a bicycle for a place at 15 miles an hour, I am one hour too soon; but if I travel on a tricycle at 10 miles an hour, I am an hour too late. How far is it?

27. Divide £367 amongst A, B, C so that B has £10 more than C, and A has £17 more than B.

28. A starts from London to walk to Brighton (50 miles) at 4 miles an hour, and B starts from Brighton 2 hours earlier at 3 miles an hour. Where do they meet, and how far will B be from London when A arrives at Brighton?

29. Write out in the senary scale the Multiplication Table (up to five times) of 453 (senary or six scale), and divide 432510 (senary) by it, proving your result by Multiplication.

30. What are the advantages and disadvantages of the binary (2) and duodenary (12) scales?

31. Illustrate your answer by expressing 63 (denary) in the binary scale and 107 in the duodenary.

32. Divide £100 amongst 24 men, 36 women, and 41 children, so that every 2 women has as much as 3 children, and every 8 men as much as 35 children. What does each man and woman receive?

33. Divide £99, 9s. amongst 20 men, 15 women, and 27 children, so that each woman may have 5s. more than a child, and each man 7s. more than a woman.

34. How much would it take to divide amongst 20 men, 15 women, and 27 children, so that each child had 5s. less than a woman, and each woman 7s. less than a man; the women receiving £5, 5s.?

35. A and B start from London for Cambridge (55 miles) on bicycles, A at 9 o'clock, B at 10. B passes A at noon and dines from 12·20 to 1, after which he proceeds at A's rate and arrives at Cambridge at 3. Whereas A only allows himself half-an-hour for dinner and does not alter his pace. Show that they arrive at same time.

36. Supposing B had neither altered his pace nor dined, what time would A have had to have taken for his dinner so as to arrive in Cambridge two hours after B.

37. Divide £108 amongst A, B, C, and D so that A has £10 more than B, and C £8 more than D, and B £5 less than C. Find B's portion without finding the others.

Note.—In question 37 ***with the result***, there are five elements, any one of which might be the thing sought for.

38. What sum of money shall I require to divide amongst A, B, C, D, if A have £10 more than B, C £8 more than D, and B, who has £24, which is £5 less than C?

39. If I divide £108 amongst A, B, C, D so that B shall have £24, A has £10 more than B, C £8 more than D. How much shall B have less than C?

40. I divide £100 amongst A, B, C, D, E, of which B has £15. The sum of A and E's shares is equal to that of B and D's, viz. twice as much as C's. Find C's and D's shares.

41. If in question 40 we gave some relationship between either A or E and any one of the others' shares, we could also find those of A and E; *e.g.* if E's share is one-fifth as great again as that of D, find the shares of A and E.

In question 40 we might use the share of either C or D as a thing known, and ask for one of the elements given, thus—

42. I divide £100 amongst A, B, C, D, E, of which B has £15 and C £20, and I know that the sum of A and E is equal to that of B and D. Show that D's share is as much greater than £15 as it is less than £65.

43. Show that the sum of the sum and difference of any two numbers is twice the greater. Illustrate (1) with numbers, as 7 and 4; (2) with lines, as 5 units and 3 units.

44. Show that the difference of the sum and difference of any two numbers or quantities of a like denomination is twice the smaller number or quantity. Illustrate (1) with 7 and 4, and (2) with a shilling and 2d.

45. Divide 99 into 3 parts, so that the first part is as much greater as the third is less than the middle part, and the third is 10 less than the first.

46. Multiply 374t by 23 in the undenary (11) scale.

47. Divide (by factors) 47622 by 34 both in the octonary scale.

48. Divide (by factors) 37269 by 31 both in the nonary scale, and show clearly how you get your remainder.

49. Reduce 3751 into its prime factors; reduce these prime factors to the septenary scale. Multiply them together in that scale and reduce your result back to the denary scale.

50. Divide 362141 (septenary) by 343 (septenary).

APPENDIX TO CHAPTER II.

In addition and subtraction of decimals place the units under units, point under point, and proceed as in integers.

In multiplication the figure obtained by multiplying the units by the units must be units; hence place the point after it, or mark as many decimal places as there are in the multiplicand and multiplier together.

In division, since any denomination divided by a number of the same denomination gives unity, mark the figure in the dividend which corresponds to the right-hand figure of the divisor, and when you have divided that figure place the point in the quotient. Do this in the course of the working.

Rules.

To reduce a number to its equivalent in another scale of notation, as say 9, divide the number by 9 again and again until the quotient is less than 9. Then the first remainder will be the unit figure, the second remainder the next figure, and so on, the final quotient being the left-hand figure.

To reduce a number expressed in any scale of notation, say 8, to the common or denary scale, multiply the left-hand figure by 8 and add in the next figure, then multiply this product by 8 and add in the third figure, and so on, and the final product will be the number expressed in the denary scale. The operation is precisely the same as reducing from a higher to a lower denomination.

To reduce from one scale (say 6) to another, not the denary (say 7), it is better to take it through the denary; but can be done at once by dividing by 7 till the quotient be less than 7; but the dividend being expressed in the senary scale, we must carry the remainder multiplied by 6 and not 10.

To reduce a decimal to a radix fraction whose radix is, say 7, multiply the decimal by 7 and mark the decimal point; the figure to the left of the point is the first figure after the unit point in the radix fraction. Now multiply the decimal of the second line and mark the decimal point, and the figure to the left of the point will give us the second figure after the unit point, and so on.

CHAPTER III.

Weights and Measures — Reduction and Four Compound Rules—Bills of Parcels—Two Kinds of Division.

1. The connection between troy weight and avoirdupois is that 7000 grains troy is equivalent to 1 pound avoirdupois. The pound troy only containing $12 \times 20 \times 24$, or 5760 grains. (It may be noticed that the last three figures are the same as those in the number of yards in a mile, viz. 1760.)

2. Since there are only 12 ounces to the pound troy and 16 ounces to the pound avoirdupois,—though the pound avoirdupois is greater than the pound troy by 1240 grains,—the ounce troy is greater than the ounce avoirdupois by $42\frac{1}{2}$ grains.

3. Reduction may be well defined as 'changing form without changing value,' and is an operation mechanically performed in working questions of the most rudimentary kind; *e.g.* in dividing 26 by 3, I change or reduce the 2 tens into its equivalent 20 units before adding it on to the 6 units.

4. It is an axiom that only quantities of the same denomination are capable of being reduced to the same denomination, can be really added together, or, to speak more correctly, amalgamated into a single quantity.

Of course 3 oranges + 4 knives are added together by the placing of the sign + between them; but they cannot be amalgamated into a single quantity, with a denomination at least signifying their real nature. All that you could do, would be to say that together they included 7 articles. This is far more important than would at first sight appear; and many a student, on entering college, is surprised at the mistakes he finds himself making, entirely owing to the want of a thorough grasp of this principle.

5. If we have to change the value of a quantity of coins of one denomination into those of another, *e.g.* coins of the value of £1, 1s. 6d. each into those of the value of 7s. 4d. each, we must reduce the one quanity into some denomination which is exactly contained in the other.

If asked how many coins, each worth 3s. 4d., would be

required to pay a bill of 75 coins, each worth 8s. each, we should proceed thus—

```
         75 eight shillings
          8
        600 shillings
          3
    10)1800 fourpences
        180 three and fourpences.
```

Here we have reduced the quantity to fourpences, since fourpence is contained an exact number of times in both 8s. and 3s. 4d. In the example above we should have to reduce the quantities named into twopences.

6. The following working of a Reduction question—both to a smaller and higher denomination—explains itself, and may be used as a model of all Reduction questions, in which the denomination ought to be inserted as far as possible in every line.

Reduce 3 ac. 3 rd. 17 po. 15 square yards to square inches, and prove your result.

```
     ac. ro. po. sq. yds.
      3   3  17   15
      4
     15 ro.
     40
    617 po.
    30¼
  18525
    154¼
  18679¼ yds.
       9
 168113¼ ft.
     144
  672452
 672452
16811336
24208308 in.
```

The 36 in the last line of the multiplication by 144 being the ¼ of 144.

144 $\begin{cases} 12)\underline{24208308} \text{ inches} \\ 12)\underline{2017359} \text{ twelve inches} \end{cases}$
9)$\underline{168113}$ ft. 3 twelve in. } 36 inches
18679 yards, 2 feet
$\underline{\quad 4}$
121 $\begin{cases} 11)\underline{74716} \text{ quarter yards} \\ 11)\underline{6792} \text{ eleven qr. yds. 4 qr. yards} \end{cases}$
4,0)$\underline{61,7}$ poles 5 eleven qr. yards } 59 qr. yards
4)$\underline{15}$ roods 17 poles
3 acres 3 roods;

	ac.	rd.	po.	yds.	ft.	in.
or,	3	3	17	$14\frac{3}{4}$	2	36;

but 2 ft. 36 in. = 1 quarter yard;
$\therefore$ $14\frac{3}{4}$ yds. + 2 ft. 36 in. = 15 yards.

7. Compound Multiplication, when the multiplier is large, may be done thus—

£379, 17s. $11\frac{3}{4}$d. × 4797.

4797
$\underline{\quad 3}$
4)$\underline{14391}$ farthings
3597d. 3 farthings
4797 × 11 = $\underline{52767}$
12)$\underline{56364}$d.
4697s. 0d.
4797 × 17 = $\underline{81549}$
2,0)$\underline{8624,6}$s.
£4312 6s.
4797 × 379 = $\underline{1818063}$
£$\underline{1822375}$

Ans. £1822375, 6s. $0\frac{3}{4}$d.

The actual working can often be diminished in difficulty by finding (1) the value of some larger quantity of a single denomination, (2) the value of the difference between this and the quantity given, and (3) subtracting. Thus—

£379, 17s. 11¾d. = £400 − £20, 2s. 0¼d.

(1) £400 × 4797 = £1918800.

(2)

$$\begin{array}{r l}
4\underline{)4797} & \text{farthings} \\
12\underline{)1199}\text{d.} & 1 \text{ farthing} \\
99\text{s. } 11\text{d.} & \\
4497 \times 2 = \underline{9594} & \\
2,0\underline{)969,3}\text{s.} & \\
£484 \;\; 13\text{s.} & \\
4797 \times 20 = \underline{95940} & \\
\underline{£96424} &
\end{array}$$

or £96424, 13s. 11¼d.

This is the value of £20, 2s. 0¼d. × 4797, we must therefore subtract this from £400 × 4797.

(3)

$$\begin{array}{r r r}
£1918800 & 0 & 0 \\
96424 & 13 & 11\frac{1}{4} \\
\hline
£1822375 & 6 & 0\frac{3}{4}
\end{array}$$

This may also be used as a method of proof.

8. In the denary scale it is not practicable to divide in one effort (see Chap. II. par. 6) by any fraction, except such as has either 2 or 5 or 10 for its denominator. In the duodenary we could divide in a single effort by fractions, with 2, 3, 4, 6, or 12 as their denominators.

Divide in one effort 37*t*46*e* (duodenary) by $4\frac{1}{3}$ (*t* stands for ten, *e* for eleven). First write out at the side in the duodenary scale the Multiplication Table of $4\frac{1}{3}$ up to eleven times. Thus—

The working then becomes easy. Let us write down the remainders underneath the corresponding figures. The remainders $\frac{1}{3}$, $\frac{2}{3}$ of course are carried as 4 and 8.

$$\begin{array}{l c c c c c c l}
4\frac{1}{3}) & 3 & 7 & t & 4 & 6 & e & \\
\hline
 & & t & 1 & 5 & 8 & 0 & \ldots 3 \\
 & & \frac{1}{3} & 1\frac{2}{3} & 2\frac{1}{3} & \frac{2}{3} & 3 &
\end{array}$$

$4\frac{1}{3}$ times

$$\begin{array}{l}
1 = 4\frac{1}{3} \\
2 = 8\frac{2}{3} \\
3 = 11 \\
4 = 15\frac{1}{3} \\
5 = 19\frac{2}{3} \\
6 = 22 \\
7 = 26\frac{1}{3} \\
8 = 2t\frac{2}{3} \\
9 = 33 \\
t = 37\frac{1}{3} \\
e = 3e\frac{2}{3}
\end{array}$$

Since the 37 is within $\frac{1}{3}$ of the number into which $4\frac{1}{3}$ goes ten times, we subtract 4 from the third figure to make it go ten times, and insert the remainder underneath in the form of minus $\frac{1}{3}$, which we write $\frac{\bar{1}}{3}$. This added (arithmetically subtracted) to t gives 6, which gives a quotient one and $1\frac{2}{3}$ remainder, so at the last figure but one we take 8 from the units to make 2t exactly divisible, and insert the remainder as $\frac{\bar{2}}{3}$.

9. In Chapter I. we have seen how to reduce a number from any scale other than the denary to another also other than the denary, doing it through the denary; but it is not necessary to do this, though perhaps practically as rapid as the more direct method. If by dividing 3476 (denary) by a succession of eights we reduce it to the octonary scale, we shall do the same if we divide 3476 (nonary) by a succession of eights. Thus—

$$\begin{array}{l} 8\underline{)3476} \text{ nonary} \\ \quad 8\underline{)387} \text{ eights}^2 \ldots 4 \text{ ones} \\ \qquad 8\underline{)44} \text{ (eight)}^2\text{s} \ldots 2 \text{ eights} \\ \qquad\quad 5 \text{ (eight)}^3\text{s} \ldots 0 \text{ (eights)}^2; \end{array}$$

therefore 3476 nonary = 5024 (octonary), both of which will be found to be 2580 denary.

10. In Bills of Parcels, I suspect if accountants could analyse the method by which they almost seem to feel the result of what appears to an unpractised mind a very difficult operation, they would tell us that they (1) split up one of the terms to be operated upon into two or more terms, and (2) that they increase one or other of the terms so as to 'round' it and then decrease the result afterwards; *e.g.* you buy 8 lbs. 7 oz. of beef at $10\frac{3}{4}$d. a lb., and before you can open your purse the butcher boy tells you to a farthing, or at any rate a halfpenny, what the price is. Now, how does he do this? (1) He will work the 8 lbs. 7 oz. as $8\frac{1}{2}$ lbs. and call the price 11d., which will give him $93\frac{1}{2}$d., or 7s. $9\frac{1}{2}$d. Now this is a farthing too much on every lb., so he subtracts 2d. for the 8 lbs., which leaves 7s. $7\frac{1}{2}$d.; there are two errors left, viz. $\frac{1}{16}$ of $10\frac{3}{4}$d. (which is more than $\frac{1}{2}$d. and less than 1d.) and also $\frac{7}{16}$ of

a farthing, both of which ought to be subtracted from this amount; he therefore subtracts another 1d., and charges 7s. $6\frac{1}{2}$d. or 7s. 7d., of which the former is $\frac{13}{64}$ of a penny too little, and the latter $\frac{19}{64}$ too much. Had he charged 7s. $6\frac{3}{4}$d. he only charged $\frac{3}{64}$d. too much.

11. Though you can add and subtract quantities of like denomination, it is not possible to multiply any quantity by anything but an abstract number. (Feet, etc., by feet, etc., may seem an exception, which will be discussed farther on.) Though I have seen, I am sorry to say, a question proposed thus. Multiply \$35·5 by \$2·575. Such an operation was an absurdity. I can take 9s. 3 times and produce 27s., but 9s. × 3s. produces nothing whatever. Now if 9s. × 3 = 27s., 27s. ÷ 3 gives 9s.; but no less does 27s. ÷ 9s. give 3, an abstract number. Hence, though it is impossible to multiply 9s. by 3s.; it is not impossible, by any means, to divide 27s. by 3s., only we must notice that in this latter case the quotient is an abstract number, and not a concrete quantity.

The sign : used in Proportion and read 'is to' really implies the division of the preceding number by the other, and always implies that they are both abstract, or of the same denomination, and in fact the expression always represents an abstract number; *e.g.* 8 : 4 represents 2 no more nor less than £6 : £3, or 10 tons : 5 tons. The expressions, then, 3 tons : 5s. or £4 : 2 are as senseless as it is possible to conceive of.

12. To find the measures of these expressions £4, 3s. 4d. : 2s. 3d. we must reduce them to the same denomination (highest possible), and divide the former by the latter. Thus—

s.	*d.*		£	*s.*	*d.*
2	3		4	3	4
12			20		
27d.			83s.		
			12		
			1000d.		

$$\begin{array}{l} \quad\;\; d.\quad d. \\ 27)1000(37\tfrac{1}{27} \text{ times} \\ \quad\;\;\; 81 \\ \quad\;\;\; \overline{\;190} \\ \quad\;\;\; 189 \\ \quad\;\;\; \overline{\quad\;\; 1} \\ \end{array}$$

that is, if I take 27 pence $37\frac{1}{27}$ times I shall obtain £4, 3s. 4d.; or if I divide £4, 3s. 4d. by $37\frac{1}{27}$, I shall obtain 2s. 3d. as my quotient.

Examination and Examples.

1. Change 3 qrs. 17 lbs. avoirdupois to troy lbs. oz., etc.

2. Change 1 cwt. 1 qr. 17 lbs. 8 oz. avoirdupois to troy lbs. oz., etc.

3. Change 3 lbs. 7 oz. 15 dwts. into lbs. avoirdupois.

4. In the five denominations of apothecaries' weight, and four of troy weight, three are the same. Compare each of the remaining three with the remaining one.

5. How many dwts. are there in 6 drs. 2 sc. 8 grs.?

6. How many coins, each worth 2s. $1\frac{1}{2}$d., are equivalent to 108, each worth 2s. 10d.?

7. How many lbs. of tea, each worth 2s. 3d., *or* lbs. of coffee, each worth 1s. 1d., can be bought for £409, 10s.? And how many lbs. of tea *and* coffee (equal quantities of each) could be purchased with the same amount?

8. How many plots of land, each containing 1 po. 1 yd. $1\frac{3}{4}$ ft., are there in a field of 5 ac. 2 rd. 17 po. 2 yds. $2\frac{3}{4}$ ft.?

9. If A takes 3 hrs. walking exercise every day (Sundays included), his average pace being $3\frac{1}{2}$ miles an hour, and B takes 2 hrs. horse exercise a day, at an average of $9\frac{3}{4}$ miles an

hour, how much more ground would B have gone over in 4 years than A.?

10. Multiply (after the method shown in par. 7) £43, 17s. $2\frac{1}{4}$d. by 935, and prove your result by Division.

11. Multiply (after par. 7) 3 ac. 2 rd. 11 po. 17 yds. 1 ft. by 51, and prove your result by Division.

12. A takes 3 hrs. walking exercise every day, and B takes 2 hrs. horse exercise every day at the average rate of $9\frac{3}{4}$ miles an hour, B goes over 13149 miles more ground than A in 4 years, how many miles an hour is A's average pace?

13. The number of the *Daily Telegraph* on Monday, October 20, 1884, was 9173, on what day of the week, month, year, etc., was No. 1 published?

14. How many Sundays are there in 504 years?

15. Given that October 19, 1884, was a Sunday, how many Sundays are there between February 29, 1880, and September 17, 1886, inclusive. Find the dates on which the Sundays will fall in March 1885.

16. How many Sundays might there be in 505 years? On what facts might the number depend?

Note.—There are 4 minutes' difference in the clock for every degree of longitude; clocks in places to the West pointing so much earlier, and those to the East so much later in the day.

17. What o'clock is it at longitude 15° W., when it is noon at longitude 0° (Greenwich)?

18. What o'clock is it at longitude 23° 30′ W., when it is 4.25 p.m. at 23° 30′ E.?

19. A ship's saloon clock is corrected at 1 p.m. each day, how much must be put back or forward at 1, if the ship has passed over 11° of longitude from (1) E. to W. (2) from W. to E.?

20. If the ship be going from W. to E., and the clock has to be put on 35′ 30″, how many degrees of longitude has she passed over?

21. How many ·09 inches can be taken from 1 fur. 5 po. 2 yds.?

22. How many ·03 oz. can be weighed from 7·0721 oz., and what is the remainder?

23. What number of lbs., oz., etc., of pure gold will there

be in 357 sovereigns, of which each contains 5 dwts. $3\frac{1}{3}$ grs.? (*N.B.*—The fraction is slightly simplified.)

24. Make out a bill for the following articles:—77 saddles at 18s. $8\frac{1}{2}$d. each, 317 sets of harness at £3, 17s. 6d. a set, 15 carriages at £19, 12s. 1d. each, and 7 coaches at £117, 4s. $8\frac{1}{2}$d. each.

Note.—As the student must work at innumerable questions of this kind to insure accuracy and rapidity, and they present no possible peculiarity in the manner of setting, no more will be given in this book, except where they are found in examination papers actually set, and which are reprinted here in full. The size of the book has prevented this.

25. If a clock gains $2\frac{1}{2}$ minutes a day, and it was exactly correct at 2 p.m. on October 16, what time will it indicate at noon on Christmas day?

26. If a clock was exactly correct at 4 p.m. on January 1, but indicated 2 hrs. 25′ at noon on June 10, what had it gained each day?

27. A clock was exactly correct at noon, February 29, but it lost $1\frac{1}{2}$ minutes each day till April 3 at noon; its regulator, but not the hands, was touched so that it gained $\frac{1}{2}$ a minute a day, what time did it indicate on June 3 at noon?

28. A clock was exactly correct at noon, February 29, but was losing $1\frac{1}{2}$ minutes a day. One day at noon the regulator was touched (the hands not being altered) so that it gained $\frac{1}{2}$ a minute a day. On June 3 at noon the clock indicated 11 hrs. 39′ 30″, find the day on which the alteration of speed took place.

Note.—In question 27 we have taken all the data (except date April 3) with the answer, and asked for April 3.

29. Similarly—writing out the question for yourself—give yourself all the data and result of 27 (except February 29), and find it.

30. Again, take all the data and result (except the $1\frac{1}{2}$ minutes), and find it.

31. Again, take all the data and result (except June 3), and find it.

32. Lastly, take all the data and result (except the $\frac{1}{2}$ minute's gain), and find it.

The proper wording of these last four questions is a part of the work to be done, and will be found in the book of answers.

33. A baker's lb. weight is $\frac{1}{2}$ oz. too light; if bread is 1d. a pound, of how much does he cheat the public, in the course of a year, if he sell every week 4257 four-pound loaves?

34. If an apothecary purchase Epsom salts at 1d. a pound avoir., and sell it at 1d. an oz. apoth., what profit does he make out of a stone of the medicine?

35. If a grocer mix 125 lbs. of tea at 2s. 2d. with 200 lbs. at 3s. 3d., and sell the mixture at 3s., what does he gain or lose by the transaction?

36. Since October 24, 1884, fell on a Friday, what day of the week was February 29, 1836?

37. Show that if a line of omnibuses start from either end of the route every four minutes, they ought to meet one of their omnibuses every 2 minutes.

38. If omnibuses travel at the uniform rate of 6 miles an hour, and start every ten minutes from either end, how many will be required to serve a journey of 6 miles long?

39. A tradesman marks his goods at the uniform profit of 4d. in a shilling, but takes off 2d. in the shilling for ready money; if, in a year, he purchases £1002 worth of goods, and receives £300 ready money payments, losing in bad debts 1s. in every pound he books, what are his profits?

40. A tradesman marks his goods as in 39; he invests a certain sum of money, and receives £300 ready money payments, and loses in bad debts a shilling in every pound booked, and makes as profit £223, 6s., what does he invest?

41. Giving yourself all the data and the result of 39, except the 4d. in a shilling, find it.

42. Secondly, find the 2d. he takes off for ready money.

43. Find the money he receives as ready money.

44. Find the amount of his bad debts.

Note.—Here again the proper wording of questions 41–44 are part of the work to be done, and will be found in the book of answers.

45. A and B buy two horses when they are colts, paying £20 a piece for them; A keeps his horse 3 years, at an average cost of 5s. a week, B keeps his for 2 years, at a cost of

6s. a week; after this A keeps his horse at a cost of 10s. a week, and earns 25s. a week with it for 12 years, whereas B keeps his still at 6s. a week, and finds he can only earn 20s. a week with it, and this kills it at the age of 12, how much more does A make out of his horse than B of his?

46. Supposing you know the result, and use it as a thing given, what elements could you find, one after the other, and what could you not find?

47. Find the age of B's horse when it began to work.

48. Find the age of B's horse when it died.

49. Find the cost of keep of B's horse?

50. Find the earnings of A's horse.

APPENDIX TO CHAPTER III.

Be sure and write the denomination at the end of every line in a Reduction question. The denomination of the quotient is obtained by reading the divisor with the denomination of the dividend.

CHAPTER IV.

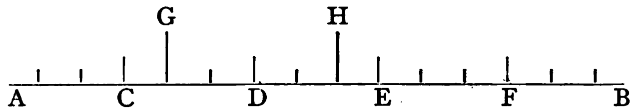
Simple Fractions and Practice—Simple and Compound.

1. It is a very good method for those working at fractions for the first time to write the denominators in words, and to look upon the reduction of fractions to other denominators as precisely the same operation as the reduction discussed in the last chapter.

2. By the definition of a half, a third, a quarter, a sixth, and a twelfth, it is evident that there are 6 twelfths in a half, 4 in a third, 3 in a quarter, and 2 in a sixth. If, therefore, I want to add together $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{12}$, $\frac{1}{2}$, $\frac{2}{3}$, writing the denominators as words and reducing to twelfths—

3 quarters = 9 twelfths
5 sixths = 10 twelfths
7 twelfths = 7 twelfths
1 half = 6 twelfths
2 thirds = 8 twelfths
sum required = 40 twelfths;
but since twelve twelfths = 1 unit,
$\therefore$ 40 twelfths = 3 units + 4 twelfths;
and since 4 twelfths = 1 third, we get as a final answer
$3\frac{1}{3}$.

3. To find $\frac{2}{3}$ of $\frac{4}{5}$, we must first divide the unit into 5 *equal* parts, and then each of these 5 parts into 3 equal parts, of which there will be 15, and therefore each of them is called a fifteenth. Let A B be our unit divided into 5 equal parts, in C D E F, and let each of these be subdivided into three equal parts—



Now A F = four-fifths of A B, and if we divide A F, which contains twelve-fifteenths, into 3 equal parts, we shall find these divisions to be at G H, when A G contains four-fifteenths, and when A H contains eight-fifteenths, hence we *see* $\frac{2}{3}$ of $\frac{4}{5} = \frac{8}{15}$.

Hence, if a rule is wanted, it is self-evident, viz. to multiply the denominators for the new denominator and the numerators for the new numerator.

4. The extreme simplicity of this rule often leads beginners to the utter absurdity of following the same course with a question in addition, thus: $\frac{3}{4} + \frac{2}{5}, = \frac{5}{9}$, which is as absurd as if one were to say 3 oranges and 2 knives make 5 Chinamen.

5. Practically to find $\frac{3}{4}$ of $\frac{5}{7}$ is the same as multiplying $\frac{5}{7}$ by $\frac{3}{4}$, and we may replace the sign 'of' by that of multiplication.

6. In an expression the signs + and − are said to rule those of × and ÷ (unless, of course, brackets distinctly show it otherwise); that is, the multiplications and divisions are supposed to be performed before the additions and subtractions.

$$\text{Thus } \tfrac{3}{4} + \tfrac{2}{3} \times \tfrac{4}{5} - \tfrac{3}{5} \div \tfrac{2}{3} + \tfrac{1}{2}$$
$$\text{means this: } \tfrac{3}{4} + (\tfrac{2}{3} \times \tfrac{4}{5}) - (\tfrac{3}{5} \times \tfrac{3}{2}) + \tfrac{1}{2},$$
$$\text{and not } (\tfrac{3}{4} + \tfrac{2}{3}) \times (\tfrac{4}{5} - \tfrac{3}{5}) \div (\tfrac{2}{3} + \tfrac{1}{2}).$$

7. It is easy to show that to obtain three-fourths of a thing is to divide it by $\frac{4}{3}$; thus to obtain the fourth of a thing we must divide it by 4, and to get three of these parts, multiply it by 3. In other words, we divide it by 4 and multiply the result by 3; but the order of operations is immaterial, hence we may, if we like, multiply first by 3 and divide the result by 4. To divide a quantity by $\frac{4}{3}$ we first of all multiply it by 3 to reduce it to thirds, and then divide it by 4 to see how many four-thirds there are in it, which is precisely the same operation as before; hence to multiply by $\frac{3}{4}$ is the same as dividing by $\frac{4}{3}$.

8. In explaining the division of a quantity by another

quantity of the same denomination, the explanation is far simpler than that of dividing a quantity by an abstract number. *E.g.*—

I divide £100 amongst 3 men and a child, giving the child one-third of the share of a man. Here I have to divide £100 by $3\frac{1}{3}$, or by $\frac{10}{3}$.

Since each man has 3 times as much as the child, the 3 men and 1 child have the same as 10 children—

Therefore $10\underline{)\pounds 100}$ / £10 will give us the share of a child,

and £10 × 3 will give each man's share, or thus—

$$\begin{array}{rl} 10)\underline{\pounds 100} & \\ \pounds 10 & \text{a child's share} \\ \underline{3} & \\ \underline{\pounds 30} & \text{a man's share.} \end{array}$$

9. In ordinary Practice questions, much time may often be saved by imitating the butcher boy as described in Chap. III. par. 10, and finding the value at the next largest 'round' sum and subtracting the value of the articles at the difference between this round sum and the value given.

We will work the same question in both ways.

To find the price of $1857\frac{1}{2}$ articles at £2, 17s. $9\frac{1}{2}$d.

		£	*s.*	*d.*
$1857\frac{1}{2}$ art. at £1	=	1857	10	0
£2	=	3715	0	0
10s. $(\frac{1}{2})$	=	928	15	0
6s. 8d. $(\frac{1}{3})$	=	619	3	4
1s. $(\frac{1}{10})$	=	92	17	6
$1\frac{1}{2}$d. $(\frac{1}{8})$	=	11	12	$2\frac{1}{4}$
		5367	8	$0\frac{1}{4}$

	£	s.	d.
$1857\frac{1}{2}$ art. at £1 =	1857	10	0
£3 =	5572	10	0
2s. $(\frac{1}{10})$ =	185	15	0
2d. $(\frac{1}{12})$ =	15	9	7
$\frac{1}{2}$d. $(\frac{1}{4})$ =	3	17	$4\frac{3}{4}$
	205	1	$11\frac{3}{4}$
	5367	8	$0\frac{1}{4}$

£	s.	d.
3	0	0
2	17	$9\frac{1}{2}$
0	2	$2\frac{1}{2}$

The difference of labour here does not seem perhaps at first sight quite as great as we might have expected. True, we have the same number of lines of work, but one of these need not have been written down, viz. the line £205, 1s. $11\frac{3}{4}$d., as it might have been subtracted from £5572, 10s. as found, and the number of figures in each line is considerably less.

Be sure and put the aliquot parts, which fraction they are, and the result in the same horizontal line, that every line of the work may read intelligibly.

Young students often fail in their papers in this respect. For instance, supposing one were asked to

Multiply 4 by 3, add on 2, divide the result by 7, and take the quotient 8 times. It is very bad arithmetic to write down—

$$4 \times 3 = 12 + 2 = 14 \div 7 = 2 \times 8 = 16;$$

for 4×3 is *not* $= 12 + 2$, and $12 + 2$ is not $= 14 \div 7$, etc.

It may seem unnecessary to take the trouble to write down the result of last operation again, but it is absolutely fatal to your thorough grasp of the science if you allow yourself to indulge in such loose and untrue statements.

The work ought to have appeared thus—

$$4 \times 3 = 12,\ 12 + 2 = 14,\ 14 \div 7 = 2, \text{ and } 2 \times 8 = 16;$$

$$\text{or, } \frac{(4 \times 3) + 2}{7} \times 8 = 16.$$

10. Compound Practice can also be often shortened by adopting the same method of first finding the value at the next 'round' number.

To find the cost of 4 tons, 12 cwts. 3 qrs. 12 lbs. at £784, 7s. 7d. per ton.

		£	s.	d.
1 ton	=	784	7	7
4 tons	=	3137	10	4
10 cwts. ($\frac{1}{2}$)	=	392	3	$9\frac{1}{2}$
2 cwts. ($\frac{1}{5}$)	=	78	8	$9\frac{1}{10}$
2 qrs. ($\frac{1}{4}$)	=	19	12	$2\frac{11}{40}$
1 qr. ($\frac{1}{2}$)	=	9	16	$1\frac{11}{80}$
7 lbs. ($\frac{1}{4}$)	=	2	9	$0\frac{91}{320}$
$3\frac{1}{2}$,, ($\frac{1}{2}$)	=	1	4	$6\frac{91}{640}$
1 ,, ($\frac{1}{7}$)	=		7	$0\frac{13}{320}$
$\frac{1}{2}$,, ($\frac{1}{2}$)	=		3	$6\frac{13}{640}$
		3641	15	$2\frac{1}{2}$

		£	s.	d.
1 ton	=	784	7	7
5 tons	=	3921	17	11
5 cwts. ($\frac{1}{4}$)	=	196	1	$10\frac{3}{4}$
1 cwt. ($\frac{1}{5}$)	=	39	4	$4\frac{11}{20}$
1 cwt. ($\frac{1}{5}$)	=	39	4	$4\frac{11}{20}$
14 lbs. ($\frac{1}{8}$)	=	4	18	$0\frac{91}{160}$
2 ,, ($\frac{1}{7}$)	=		14	$0\frac{13}{160}$
		280	2	$8\frac{1}{2}$
		3641	15	$2\frac{1}{2}$

In which the saving of time is considerable; and here too the writing of the shorter method might be reduced by the line £280, 2s. $8\frac{1}{2}$d.

An aliquot part of a quantity is one that is contained an exact number of times in the quantity.

Examination and Examples.

1. Write down all the aliquot parts of a £ from 10s. to a farthing.

2. Write down all the aliquot parts of 6s. 8d. from 3s. 4d. to a farthing.

3. Write down all the aliquot parts of a ton from 10 cwts. to 2 quarters.

4. Write down all the aliquot parts of an acre from 2 roods to 11 yards.

5. Find both ways, as shown in par. 9, the value of $3245\frac{3}{4}$ articles at £1, 19s. 1d. each.

6. How many twenty-fourths are there in a half, a third, a quarter, a sixth, an eighth, and a twelfth? Reduce two-thirds, three-fourths, five-sixths, three-eighths, and one-twelfth to twenty-fourths, add them together, and reduce the sum to units.

7. Find in two lines the value of 375 tons at £5, 17s. 6d. a ton.

8. Find (only using two aliquot parts) the value of $1004\frac{2}{3}$ lbs. at £2, 17s. 0d. a lb.

9. Find (only using two aliquot parts) the price of 4 tons, 19 cwts. 3 qrs. 21 lbs. at £5, 6s. 8d. a ton.

10. What would be the easiest aliquot parts to take for 2s. $2\frac{2}{3}$d.?

11. Multiply £3, 4s. $7\frac{1}{3}$d. by 7.

12. Divide £3, 4s. $7\frac{1}{3}$d. by $7 \times 3 \times 5$.

13. Multiply 5 po. 2 yds. $1\frac{1}{3}$ ft. by 4×7.

14. Divide 5 sq. po. 17 yds. 2 ft. by 5×7.

15. Multiply $342\frac{1}{3}$ septenary by 5.

16. Divide $4213\frac{2}{3}$ senary by 7.

17. Divide $53217\frac{2}{3}$ nonary by $3 \times 7 \times 2$.

18. Divide $4t\ 5e\ 3\frac{3}{t}$ duodenary by $t \times 3 \times 7$.

19. Divide into five equal parts 4 ac. 2 po. 2 in.

20. Find by Practice the value of 4 ac. 2 po. 1 in. at 1s. a square rood.

21. Reduce to its primary factors 3315 septenary.

22. How can you tell at a glance that the above number 3315 (septenary) is even, and will divide by 6?

23. Reduce to its primary factors 4741 nonary.

24. How can you tell at a glance that this number is even, and will divide by both 8 and 10 without remainder?

25. Reduce $5214\frac{1}{3}$ octonary to twelfths expressed in the duodenary scale.

26. How many times is 657 octonary contained in 78121 nonary? Express the quotient (fractions, etc.) in the nonary scale.

27. If a sovereign weighs 5 dwts. $3\frac{1}{2}$ grs., what is the weight of 1256 sovereigns?

28. How many sovereigns could a man carry, who is capable of carrying 185 lbs. avoirdupois.

29. What is the difference between $\frac{1}{7}$ and $\frac{1}{8}$?

30. AB is a ruler divided into inches and ninths of an inch. CD is another ruler divided into inches and elevenths of an inch. If the inches, etc., are marked from A and C respectively, how far is it from the first subdivision in AB to the first in CD, and from the second to the second?

31. How many more times is 3s. 4d. contained in £5, 6s. 8d. than 5s. 4d.?

32. What number must I subtract from a million to make the remainder contain 315 exactly 417 times?

33. Find the remainders (without dividing) after 471321 has been divided by all the numbers from 2 to 12 inclusive.

34. Show (without dividing) that the number 133056 is divisible by 792.

35. Write down four consecutive numbers, of which 37 is one, and without multiplying them show that their product must be a multiple of 1, 2, 3, 4, or 24.

36. Show that every number is either a multiple of 3 or one more or one less than a multiple of 3.

37. By means of a figure, show that every square number must be either a multiple of 4 or one more than a multiple of 4.

38. Write down all the numbers between 31 and 69 that will divide by 5 and leave a remainder 2.

39. Write down all the numbers between 101 and 151 that will divide by 11 and leave a remainder 8.

40. A man undertakes to walk 40 miles, ride 40 miles, and drive 40 miles in the 24 hours. He is to receive 10s. for every mile he walks, 5s. for every mile he rides, and 2s. 6d. for every mile he drives; paying double for those he fails in. He determines to walk 8 miles, and then drive 8 miles, and then ride 8 miles, and in this order he goes on until he breaks down. If he only received £2, where did he break down?

41. Supposing we did not learn the order of his going, in what terms could we determine his position generally, according to the amount of money he received?

42. Supposing he elected to do all his walking first, and then to ride, and lastly to drive, and received £14, where did he break down?

43. Suppose he rode first and then walked, but was unable to drive at all, what did he receive?

44. He walked, drove, and rode exactly the same number of miles each, but received nothing. How far did he walk, ride, etc.?

45. At 9 a.m. on Christmas day, 1882, two clocks are put right, the one loses $\frac{1}{4}$ second an hour, and the other gains 1 minute a week, what time on Christmas Eve, 1883, does the losing clock mark when the gaining clock is at 9 hrs. 52 mins. a.m.?

46. Show that with 4d., 3d., and 6d. pieces I can pay any number of pence. How would you pay 1d.?

47. I have as weights one 1 lb., one 3 lb., one 9 lb., and one 27 lb. Show how I can weigh any number of lbs. up to 40 with these weights alone; *e.g.* how would you weigh 38 lbs. or 25 lbs.?

48. On a farm of 600 acres, of which I pay 35s. an acre rent, and one quarter of this in rates and taxes, I keep 5 men at 15s. a week. I sow the whole with grain, which costs me to harvest 81s. 9d. per acre, for which I receive an average of 32s. a quarter. How many quarters must I reap to the acre to earn an income of £600?

49. By how much is it better to have an income of £148 with no income tax, than one of £151 with an income tax of 5d. in the pound?

CHAPTER V.

Terms, Factors, Greatest Common Measure, Least Common Multiple, Reduction of Fractions, and Four Rules in Vulgar Fractions.

1. When two or more numbers are connected together by the sign of + or −, they are called terms of the expressions. In arithmetic we so seldom express numbers in this way that we hear very little about terms therein.

2. When two or more numbers are connected together by the sign of × they are called factors of the expression; thus in $6=2\times 3$, 2 and 3 are the factors of the expression 2×3, and therefore of 6.

3. When a number has no factors but itself and unity, it is called a prime number. Thus 7, 11, 13 are prime.

4. When two numbers have no common factor, as 9 and 16, they are called prime to one another.

5. Though many of the theorems connected with numbers cannot be *universally* proved without the aid of Algebra, many of them can be seen and understood without that aid.

6. An even number is one that will exactly divide by 2, and an odd number one that will always leave a remainder 1 after it has been divided by 2. You can easily see that the doubles of odd numbers are numbers that will always leave a remainder 2 after being divided by 4.

7. A measure is only another name for a factor, but is generally defined as a number that will exactly divide the number without remainder. Thus 4, 8, 12 are all measures of 24. The measures of a concrete quantity are called its aliquot parts. A common measure of two or more numbers is a number that will divide the numbers without remainder, as 6 is a common measure of 12 and 18. The greatest common measure is the greatest number that will divide two or more numbers without remainder, as 12 is the G. C. M. of 24 and 36.

Here is a little theorem as to numbers that the student will readily see. If 12 be the G. C. M. of 24 and 36, 2 and 3, the quotients, after dividing 24 and 36 by 12, must be prime to one another.

8. To find the G. C. M. of 2849 and 5328, the usual way is that found on the left-hand side. That on the right-hand side is a better method, where the divisions are performed backwards and forwards, the last divisor not being written down again.

```
2849)5328(1                                   2849      5328
     2849                                     2479      2849
     ----                                     ----      ----
     2479)2849(1                               370      2479
          2479                                 259      2220
          ----                                 ---      ----
           370)2479(6                          111       259
               2220                            111       222
               ----                            ---       ---
                259)370(1                                 37
                    259                   Method 2.      ---
                    ---
                    111)259(2
                        222
                        ---
Method 1.                37)111(3
                            111
                            ---
```

In both of which we see 37 is the G. C. M.

But even this second method may be considerably shortened. Since all that we are aiming at is to find the greatest factor common to the quantities whose G. C. M. is to be found, any factor that is contained in the one and not in the other may be eliminated at any part of the operation.

Again we will perform the operation twice,—first, expressing the factors to be eliminated, the quotients of the division, and rewriting the last divisor as the new dividend; secondly, amitting them all—

11)2849	8)5328	2849	5328
259	6)666	259	666
	3)111	259	111
Method 3.	37)259(7	Method 4.	37
	259		

We will work out two more examples, in the former of which we see at once a common factor 6 and then of 8.

9984	16848	3731	4059
1664	2808	369	451
208	351	41	41
195	39		
13	39		

G. C. M. $= 6 \times 8 \times 13$, or 624. G. C. M. 41.

As a specimen of a problem in G. C. M., we might be asked the largest coin with which it were possible to pay both of two accounts, *e.g.* £1, 2s. 9d. or 273d., and £2, 0s. 10d. or 490d.

```
273    )490(5
 91     455
 91      35        Ans. a coin worth 7d.
          7
```

9. The proof of the G. C. M. depends upon the following theorem, viz. that if a number will exactly measure, *i.e.* divide, two numbers, it will measure the sum or difference of any multiples (the reciprocal of measures) of these numbers. (456×1, or 456, is considered as a multiple of 456.) Let 4×17 and 11×17 be any two numbers. They have the common factor 17, then any multiple of 4×17 (say $3 \times 4 \times 17$) + any multiple of 11×17 (say $6 \times 11 \times 17$) is equal to some number of seventeens, or, in other words, 17 will measure it.

10. Let us find in the old way the G. C. M. of 273 and 390, and examine the operation step by step, beginning at the end,—

$$
\begin{array}{l}
273\,)\,390\,(\,1 \\
273 \\
\hline
117\,)\,273\,(\,2 \\
234 \\
\hline
39\,)\,117\,(\,3 \\
117 \\
\hline
\end{array}
$$

From this we see (i.) $117 = 3 \times 39$.
(ii.) $273 = 2 \times 117 + 39$.
(iii.) $390 = 273 + 117$.

First to prove that 39 is a common measure of 273 and 390.

From (i.) $117 = 3 \times 39$; $\therefore$ 39 is a measure of 117.

" (ii.) $273 = 234 + 39$
$= 2 \times 117 + 39$
$= 6 \times 39 + 39 = 7 \times 39$; $\therefore$ 39 " 237.

" (iii.) $390 = 273 + 117$
$= 7 \times 39 + 3 \times 39$
$= 10 \times 39$; $\therefore$ 39 " 390;

$\therefore$ 39 is a common measure of 237 and 390. And it is the G. C. M. of 237 and 390. For any greater number that divides them both will, by the theorem proved in par. 9, divide 117; and therefore 39, or a greater number, will exactly divide a less, which is impossible.

11. We have in par. 9 called a multiple the reciprocal of a measure; that is, 2 being a measure of 6, 6 is a multiple of 2. To find the L. C. M. of two numbers, say 51 and 85. What we want to do is to find the least number that contains them both as measures or factors.

Their G. C. M. will be found to be 17.

Since $51 = 3 \times 17$, and $85 = 5 \times 17$.

Since 17 is their G. C. M., 3 and 5 must be prime to each other, for had they a common factor, 17 would not be the G. C. M. of 51 and 85.

The least number, therefore, that contains both $5 \cdot 17$ and $3 \cdot 17$ must be $5 \cdot 3 \cdot 17$.

Now I can write this in this form $\frac{5.3.17.17}{17}$, or $\frac{85 \times 51}{17}$, that is, the product of the two numbers divided by their G. C. M.; or I can write it $5.17 \times \frac{3 \times 17}{17} = 85 \times \frac{51}{17}$, or the product of one of the numbers and the quotient after dividing the other by their G. C. M.

12. The old method of finding the L. C. M. of a lot of numbers was thus—

$$\begin{array}{r|cccccccccc}
2) & \cancel{2} & \cancel{4} & \cancel{6} & \cancel{8} & 10 & 14 & 18 & 21 & 24 & 27 \\
\hline
 & & & & 3) & 5 & \cancel{7} & \cancel{9} & 21 & 12 & 27 \\
\hline
 & & & & & 5 & & & 7 & 4 & 9
\end{array}$$

L. C. M. $= 2 \times 3 \times 5 \times 7 \times 4 \times 9 = 7560$.

We first eliminated the 2, 4, 6, 8, because a multiple of 24 must be a multiple of 2 or 4, or 6 or 8. We then divided by 2 (being a common factor of two of the numbers); since 2×5, 2×7, 2×9 only require one 2 as a factor to ensure their presence as a factor in the result, we rewrite the 21 and 27. Since they also have to appear in the common multiple at last, we next eliminate the 7 and the 9. Since a number containing a certain number of 21s. must also contain the same number of 7s., we then divide by 3 as a common factor of 21 and 27 for the same reason as we divided by 2 above; and we can see that $2 \times 3 \times 5 \times 7 \times 4 \times 9$ must contain as a factor all the numbers originally proposed.

13. The method we propose is as follows:—Resolve each number as it comes into its prime factors, and put them down one after the other; only, of course, inserting those factors for each number that have not been inserted before. We will write the number which necessitates the entry of each factor over that factor or factors, drawing a line between the number and the factors it introduces. Thus—

$$\begin{array}{lcccccccc}
\text{Numbers,} & 4 & 14 & 16 & 18 & 21 & 24 & 33 & 27 \\
\text{Factors,} & 2 \times 2 \times & 7 & \times 2 \times 2 \times & 3 \times 3 & & & \times 11 \times & 3 .
\end{array}$$

It will be noticed that no new factor is introduced under 21 or 24, since these numbers 3×7 and $3 \times 2 \times 2 \times 2$ are already expressed in the form of factors.

To take another example.

Find L. C. M. of 24, 36, 45, 63, 84, and 108.

$$\frac{24}{2\times2\times2\times3}\times\frac{36}{3}\times\frac{45}{5}\times\frac{63}{7}\times\frac{84}{\ }\times\frac{108}{3};$$

$$\therefore \text{L. C. M. } 2\times2\times2\times3\times3\times5\times7\times3=7560.$$

Of course, in multiplying these I first take away 2×5, or 10, and multiply the remaining figure and add a cipher for the 10.

To give one more example, without inserting the numbers which necessitate the introduction of each factor.

Find the L. C. M. of 14, 21, 25, 60, 72, 77, 99, 100.

N.B.—Where a dot (.) cannot possibly mean a unit point, it is used as a sign of multiplication.

L. C. M. 2.7.3.5.5.2.2.3.11. = 138600.

14. To find the L. C. M. of large numbers it is always better to find their G. C. M.

E.g. find the L. C. M. of 1017, 1469. Their G. C. M. will be found to be 113 and $1017 \div 113 = 9$; $\therefore$ L. C. M. $= 1469 \times 9 = 13221$.

If we know that a number is exactly divisible by another, which of course we always do when we have found the G. C. M. of two or more numbers, we can generally write the quotient down (unless there be more than two figures in it) at once by examining the first few figures and the last.

E.g. I know 1469 is exactly divisible by 113. By looking at the first 3 figures I know the first figure of quotient must be 1, and by looking at last, since nothing times 3 but 9 times 3 ends in 7, I also know the second figure must be 9.

15. Any fraction at any time may be reduced, that is, have its form altered, by multiplying or dividing both numerator and denominator by the same number. For if I multiply the denominator by any number, say 6, I change them into a denomination of which there are six in every one of the others, and therefore there must be six times as many; so just as 6s. = 72d., because there are 12 pence in every shilling, so

$\frac{3}{4}=\frac{18}{24}$, because there are six twenty-fourths in every fourth. If I can change $\frac{3}{4}$ in $\frac{18}{24}$, I can of course change $\frac{18}{24}$, back into $\frac{3}{4}$.

16. If we multiply a fraction, say $\frac{3}{4}$, by 4 or 8, or any number containing 4 as a factor, it is evident that the result must be an integer. The operation must come to this, $3 \div 4 \times$ some integer, and to divide by 4 and then multiply result by 4 can have no effect upon a number, hence the product must be an integer. This fact can be used with great advantage in simplifying what are called complex fractions, that is, where fractions appear in the numerator or denominator of a fraction; *e.g.*—

$$\frac{3\frac{1}{2}}{4}=\frac{7}{8};\ \frac{2}{3\frac{1}{3}}=\frac{6}{10}=\frac{3}{5};\ \frac{2\frac{1}{2}}{3\frac{1}{3}}=\frac{15}{20}=\frac{3}{4}.$$

In the first of these three examples we multiplied both numerator and denominator by 2, in the second we multiplied both by 3, and in the last we multiplied both by 6, as containing both the denominators 2 and 3 found in the fraction. Such expressions as the following are very easily simplified by this method; beginning at the bottom of the numerator (if it have fractions) or denominator, or even in some cases beginning at the bottom of both at the same time. Thus simplify—

$$\text{(i.)}\ \frac{1}{2\frac{3}{4\frac{1}{5\frac{1}{6}}}} \qquad \text{(ii.)}\ \frac{3\frac{2}{2\frac{1}{5\frac{1}{4}}}}{\frac{5}{7}} \qquad \text{(iii.)}\ \frac{2\frac{1}{2\frac{2}{3\frac{1}{2}}}}{3\frac{4}{5\frac{1}{1\frac{1}{2}}}},$$

$$\text{(i.)}\ \frac{1}{2\frac{3}{4\frac{1}{5\frac{1}{6}}}}=\frac{1}{2\frac{3}{4\frac{6^a}{31}}}=\frac{1}{2\frac{93^b}{130}}=\frac{130}{353}.$$

(*a*) To get this we multiplied both numerator and denominator of $\frac{1}{5\frac{1}{6}}$ by 6.

(*b*) To get this we multiplied both numerator and denominator of $\frac{3}{4\frac{6}{31}}$ by 31.

$$\frac{3\frac{2}{2\frac{1}{5\frac{1}{4}}}}{\frac{5}{7}}=\frac{3\frac{\cancel{2}1}{1\cancel{2}\frac{\cancel{4}2}{21}}^{a}}{\frac{5}{7}}=\frac{3\frac{21}{23}}{\frac{5}{7}}=\frac{90}{\frac{115}{7}}=\frac{18}{\frac{23}{7}}=\frac{126}{23}.$$

(*a*) We here first divide the numerator and denominator of $\frac{2}{2\frac{4}{21}}$ by 2 before multiplying by 21. The fraction $\frac{3\frac{21}{23}}{\frac{5}{7}}$ could have been reduced to a simple fraction (one whose numerator and denominator are both integers) in one effort by multiplying at once by 7×23, or 161; but the multiplying a mixed number, as $3\frac{21}{23}$, by a number containing 23 is not so easy, therefore it is generally better to separate the two operations.

17. In reducing a fraction to its lowest (surely it ought to be highest,—is not $\frac{1}{2}$ greater than a $\frac{1}{12}$?) terms, the rules to test the divisibility of numbers, given in Chapter I., will be invaluable.

18. In addition of fractions do not reduce the integers to the denomination of the fractions they are connected with, but add up the fractions, reduce the sum to integers, and carry these integers on to the whole numbers.

The subjoined is a very good method of adding fractions for beginners. The common denominator being written in its factorial form to the right of the numerator, the new numerator alone being inserted in connection with the fractions they represent.

Add $2\frac{3}{4}$, $15\frac{5}{6}$, $17\frac{1}{9}$, $8\frac{7}{12}$, $11\frac{13}{24}$, $\frac{7}{16}$, place them one underneath the other.

$$\left.\begin{array}{rr} 2\frac{3}{4} & 108 \\ 15\frac{5}{6} & 120 \\ 17\frac{1}{9} & 16 \\ 8\frac{7}{12} & 84 \\ 11\frac{13}{24} & 78 \\ \frac{7}{16} & 63 \\ \hline \underline{\underline{56\frac{37}{144}}} & \underline{\underline{469}} \end{array}\right\} 2.2.3.3.2.2 = 144.$$

$$\begin{array}{r} 144)469(3 \\ 432 \\ \hline \underline{37} \end{array}$$

If you keep the L. C. M. in its factorial form, the necessary divisions by the respective denominators are all self-evident; *e.g.* how many 12s. are there in 2.2.3.3.2.2? Cover up the factors that make 12, viz. 2.2.3, and the remaining 2.2.3 must be the quotient, by which you have to multiply the 7 in the fourth fraction.

To take a more difficult example, add together $\frac{2}{3}$ of $\frac{7}{13}$, $131\frac{1}{11}$, $215\frac{3}{17}$, $4\frac{1}{2} \times \frac{5}{17}$, $3\frac{1}{2} \times \frac{7}{11}$, $2\frac{1}{3\frac{1}{4}}$.

$$\left.\begin{array}{lcrr} \frac{2}{3} \text{ of } \frac{7}{13} & = & \frac{14}{3.13} & 28.187 \\ 131\frac{1}{11} & = & 131\frac{1}{11} & 39.34 \\ 215\frac{8}{17} & = & 215\frac{8}{17} & 3.39.22 \\ 4\frac{1}{2} \times \frac{5}{17} & = & 1\frac{11}{2.17} & 39.121 \\ 3\frac{1}{2} \times \frac{7}{11} & = & 2\frac{5}{2.11} & 57.65 \\ 2\frac{1}{3\frac{1}{4}} & = & 2\frac{4}{13} & 132.34 \end{array}\right\} 3.13.11.17.2.$$

We have purposely not multiplied out the L. C. M. of the denominators, as the product is not wanted till later in the operation; and we have also put down the numerators in the form of factors.

19. In Subtraction, if we have to subtract a larger number from a smaller, we must add on a unit in the form of $\frac{8}{8}$ or $\frac{17}{17}$, whatever the denominator of the fraction may be, to the minuend and as a unit to the subtrahend, thus—

Subtract $3\frac{1}{2\frac{1}{3\frac{1}{4}}}$ from $14\frac{1}{5}$.

$$\left.\begin{array}{lr} 14\frac{1}{5} \qquad\qquad\qquad\qquad 14\frac{1}{5} & 6 \\ 3\frac{1}{2\frac{1}{3\frac{1}{4}}} = 3\frac{1}{2\frac{4}{13}} = 3\frac{13}{30} & 13 \end{array}\right\} 5.2.3.$$

$$\underline{10\frac{23}{30} \quad 23}$$

Here we have to add $\frac{30}{30}$ to minuend and 1 unit to subtrahend. Be sure not to write down equal to (=) between $14\frac{1}{5}$ and 6, etc.; you often see such an untrue statement as $14\frac{1}{5}=\frac{6}{30}$.

20. Though decimals are very easily added and subtracted, no reduction being necessary, Multiplication and Division is generally better done by Vulgar Fractions, when you may cancel.

Cancelling really depends upon the fact that if a number is multiplied by any number and the result divided by the same number the second result will be the original number; *e.g.* $6 \times 4 = 24$, and $24 \div 4 = 6$. In cancelling a factor which occurs in the numerator with one that occurs in the denominator, you are simply preventing yourself performing this useless operation.

21. All questions in Multiplication and Division ought to come out in fractions whose terms are prime to one another.

22. If concrete quantities occur in a fraction, they can only be of a like kind or capable of being reduced to a like kind, and must be so reduced, when the common denomination can be cancelled as if it were a factor. Thus—

$$\frac{\pounds 2,\ 9\text{s.}}{\pounds 9,\ 9\text{s.}} = \frac{49\text{s.}}{189\text{s.}} = \frac{49}{189} = \frac{7}{27}; \text{ though } \frac{6}{7}\text{s.} = \frac{6}{7} \text{ of a shilling,}$$

$\frac{7}{4\text{s.}}$ is nonsense, as an abstract number cannot be divided by a concrete quantity.

23. A fraction can always be changed into another fraction whose numerator or denominator is given. Change $\frac{3}{5}$ into a fraction whose denominator is 7. We can make the denominator 1 by dividing both numerator and denominator by 5, and we can make this 1 seven by multiplying both numerator and denominator by 7. Our operation then will be thus—

$$\frac{3}{5} = \frac{\frac{3}{5}}{1} = \frac{4\frac{1}{5}}{7}.$$

All questions of profit and loss can be done in this way.

They are in reality such questions as these: reduce $\dfrac{2s.\ 8d.}{1s.\ 5d.}$ to a fraction whose denominator is 100.

$$\frac{2s.\ 8d.}{1s.\ 5d.} = \frac{32}{17} = \frac{\frac{32}{17}}{1} = \frac{\frac{3200}{17}}{100} = \frac{188\frac{4}{17}}{100}.$$

To take one more example. Reduce $3\dfrac{2}{7\frac{1}{2}}$ to a fraction whose numerator is $\frac{3}{5}$.

$$3\frac{2}{7\frac{1}{2}} = 3\frac{4}{15} = \frac{64}{15} = \frac{1}{\frac{15}{64}} = \frac{\frac{3}{5}}{\frac{3 \cdot \cancel{15}\,3}{64 \cdot \cancel{5}}} = \frac{\frac{3}{5}}{\frac{9}{64}}.$$

24. In such a question as this: What fraction must be subtracted from $11\frac{3}{7}$ of $2\frac{1}{4}$ so that $\frac{3}{5}$ of the difference may be greater by $\frac{2}{7}$ than the fraction $4\dfrac{2}{3\frac{1}{7}}$? refer to Chapter II. par. 1, and work the same question in simple numbers. The working of this would be as follows:—

$$4\frac{2}{3\frac{1}{7}} + \frac{2}{7} = 4\frac{\cancel{14}\,7}{\cancel{22}\,11} + \frac{2}{7} = 4\frac{49+22}{7\times 11},$$

$4\dfrac{71}{77} = \dfrac{379}{77}$ (the fraction greater by $\dfrac{2}{7}$ than $4\dfrac{2}{3\frac{1}{7}}$) which gave us

$$\frac{379}{77} \div \frac{3}{5} = \frac{379}{77} \times \frac{5}{3} = \frac{1895}{231};$$

the difference between $11\frac{3}{7}$ of $2\frac{1}{4}$ and the fraction required, the fraction therefore will be found by subtracting this difference from $11\frac{3}{7}$ of $2\frac{1}{4}$—

$$11\frac{3}{7} \text{ of } 2\frac{1}{4} - \frac{1895}{231} = \frac{\overset{20}{\cancel{80}}}{7} \text{ of } \frac{9}{\cancel{4}} - \frac{1895}{231}$$

$$= 25\frac{5}{7} - 8\frac{47}{231} = 17\frac{1155 - 329}{7.231} = 17\frac{826}{1617}$$

$$= 17\frac{118}{231}.$$

To test this answer. If it is correct, then

$$\left\{\left(11\frac{3}{7} \text{ of } 2\frac{1}{4}\right) - 17\frac{118}{231}\right\}\frac{3}{5} = \frac{2}{7} + 4\frac{2}{3\frac{1}{7}};$$

$$\text{or, } \left\{\left(\frac{\overset{20}{\cancel{80}}}{7} \text{ of } \frac{9}{\cancel{4}}\right) - 17\frac{118}{231}\right\}\frac{3}{5} = \frac{2}{7} + 4\frac{\overset{7}{\cancel{14}}}{\underset{11}{\cancel{22}}};$$

$$\text{or, } \left\{25\frac{5}{7} - 17\frac{118}{231}\right\}\frac{3}{5} = 4\frac{22 + 49}{7.11};$$

$$\text{or, } \left\{8\frac{1155 - 826}{7.231}\right\}\frac{3}{5} = 4\frac{71}{77};$$

$$\text{or, } \left\{8\frac{47}{231}\right\}\frac{3}{5} = 4\frac{71}{77};$$

$$\text{or, } \frac{1895}{3 \times 77} \times \frac{3}{5} = 4\frac{71}{77};$$

$$\text{or, } \frac{379}{77} = 4\frac{71}{77},$$

which it will be found to be.

25. In working fractions, if one sees the factors of a number it is often wise to change the number into its factors, as we may often find as factors high prime numbers which we should not otherwise have detected; *e.g.* supposing we had

the fraction $\frac{296}{333}$, we see 8 is a factor of the numerator and 9 of the denominator, let us then express the fraction thus: $\frac{8 \times 37}{9 \times 37}$, and we immediately find 37 as a common factor which may be cancelled.

RULES.

In finding the G. C. M., place the numbers side by side with a line between them. Remove at any time of the operation any factors that can be seen in either of them. If a factor appear in both it is a factor of the G. C. M. The last divisor need not be rewritten, but can be divided in the position it has, as the quotients are not required, but only the remainders.

To find the L. C. M., reduce each number to its prime factors, and see that all the prime factors in every number appear in the L. C. M.

In reducing fractions to the same denomination, it is better to keep the L. C. M. of their denominators in a factorial form.

In reducing complex fractions, multiply numerator and denominator by the L. C. M. of the denominators.

To compare fractions. Either reduce them to the same denomination, or reduce all the numerators to 1 and compare the denominators.

To add or subtract fractions. Reduce them to the same denominations and add or subtract the numerators.

To multiply fractions. Reduce them to fractions with integers in all the numerators and denominators, and after cancelling any factors common to a numerator and denominator, multiply the numerators for the new numerator and the denominators for the new denominator.

To divide by a fraction. Invert it, and proceed as in multiplication.

To reduce fractions to others with a given numerator or denominator. First make the numerator or denominator 1 by division, of course dividing both parts so as not to alter its value, and then multiply both numerator and denominator by

the number to which the numerator or denominator is to be reduced.

To find the nature of the remainder after finding how many entire times one fraction is contained in another. Reduce them to the same denomination and divide the greater by the less, the remainder will be of the denomination that the dividend is, and the quotient will be abstract.

Examination and Examples.

1. When is it better to work in decimals and when in vulgar fractions, and why?
2. Give some proof that the value of a fraction remains unaltered, if the numerator and denominator be either multiplied or divided by the same number.
3. Define an even and an odd number, and prove that the sum or difference of two odd numbers must be even.
4. Show that one of three consecutive numbers must divide by 3, and that the product of any three consecutive numbers must divide by $1 \times 2 \times 3$, or 6.
5. Define a prime number, and write out all the prime numbers from 11 to 101.
6. On what theorem of numbers does the proof of the G. C. M. depend?
7. Show that every number (except the quotients) that appears in the working of a G. C. M. question must be a multiple of the G. C. M. found.
8. Show that the L. C. M. of two numbers is the quotient of their product divided by their G. C. M.
9. Find the L. C. M. of 1703 and 2227.
10. Find the L. C. M. of 1963, 2567, 3171.
11. Find, by method given in par. 8, the G. C. M. of 6439, 8131.
12. If 7429 be the L. C. M. of two numbers of which 323 is one, and 17 their G. C. M., find the other.
13. Find the prime factors of 2475 and of 37224, also of 962.

14. What fraction must I add to $\frac{1}{2567}$ to make $\frac{17}{2703}$?

15. What does the sum of $\frac{1}{3}$ and $\frac{3}{5}$ and $\frac{6}{7}$ and $\frac{7}{9}$ and $\frac{9}{11}$ lack of 5?

16. Remembering that the signs of + and − rule those of × and ÷, find the value of $\frac{4}{5}$ of $3\frac{1}{3} + 2\frac{1}{5} \times \frac{4}{33} - \frac{1}{2} \div \frac{5}{4} + 2\frac{3}{3\frac{1}{3}}$.

17. Reduce to their lowest terms—

$$\text{(i.)}\ \frac{£3,\ 4s.\ 7d.}{£5,\ 9s.\ 4\frac{1}{2}d.} \qquad \text{(ii.)}\ \frac{£3,\ 16s.\ 6d.}{£8,\ 0s.\ 8d.}$$

$$\text{(iii.)}\ \frac{7\ \text{lbs.}\ 2\ \text{oz.}\ 1\ \text{dwt.}\ 4\ \text{grs.}}{9\ \text{lbs.}\ 7\ \text{oz.}\ 0\ \text{dwt.}\ 12\ \text{grs.}}$$

What is the nature of these results?

18. Find the least sum of money which contains an exact number of times 2s. 3d., 3s. 4d., 4s. 2d., and 6s. 9d.

19. What fraction multiplied by $2\frac{3}{4\frac{1}{5}}$ will produce $3\frac{4}{5\frac{1}{5}}$?

20. Find the G. C. M. and L. C. M. of $131\frac{2}{3}$ and $417\frac{2}{9}$.

21. In finding the L. C. M. of fractions, why need you not take the denominators into consideration?

22. I have a cask full of brandy, I draw out a quarter of it and fill it up with water, I draw out a quarter of this and again fill up with water, I do this altogether four times. If the brandy now were separated from the water, how much of the cask would it fill?

23. How many $\frac{3}{4}$ of a pint are there in 63 galls. 3 qts.?

24. How many times is $2\frac{1}{4}$ of a square foot contained in 3 rd. 5 po. 7 yds.?

25. What fraction divided by $2\frac{1}{4}$ is greater by $\frac{2}{3}$ than 3 times $4\frac{1}{7\frac{1}{5}}$?

26. Reduce $\frac{3}{5}$ to a fraction whose numerator is 108.

27. Reduce 4 to a fraction whose numerator is 7.

28. Reduce $4\dfrac{1}{7\frac{1}{2}}$ to a fraction whose denominator is 100.

29. Reduce $\dfrac{2\frac{4\frac{1}{2}}{12}}{5\frac{2\frac{1}{4}}{12}}$ to a fraction whose denominator is 100.

30. Reduce $\dfrac{4\text{s. }6\text{d.}}{5\text{s. }8\text{d.}}$ to a fraction whose numerator is 4 lbs.

31. A man leaves $\frac{1}{3}$ of his property to his wife, $\frac{1}{12}$ each to 5 daughters, and the rest to his only son. What part has the son?

32. If $13\frac{1}{2}$ sq. in. in each tile is left uncovered by the tile above it, how many tiles will be required to cover a roof of 40 sq. yds. 6 ft.?

33. Reduce $\frac{4815}{379}$ seconds to hours (only using hours).

34. Reduce $\frac{3641}{411}$ sq. ft. to poles (only using poles).

35. A young man's income increases each year by $\frac{1}{5}$ of the original income. In the first year he saves $\frac{7}{30}$ of his income, in the second $\frac{1}{5}$, and in the third $\frac{1}{4}$, but in the fourth year, from illness, he has to spend $\frac{1}{2}$ of his savings in addition to his whole income, and at the end of the year he loses his income altogether. How long could he live on the rest of his savings at half the rate of the second year's expenditure?

36. If the answer to last question is 13 months, give yourself it, and all the other elements except the portion of his savings that he spent in the fourth year, and find it.

37. As in question 35, find the rate of expenditure after he has lost his income.

38. Why would it be very hard to find the rate of income's annual increase without the aid of Algebra?

39. Find the fraction which subtracted from $4\frac{2}{7}$ of $\frac{7}{15}$ leaves a remainder which is less by $5\frac{3}{5}$ than $7\dfrac{1}{6\frac{1}{7}}$.

40. Simplify $\dfrac{\frac{2}{3}+\frac{3}{4}}{\frac{2}{3}\text{ of }\frac{3}{4}} \div \dfrac{\frac{3}{4}-\frac{2}{3}}{\frac{3}{4}\div\frac{2}{3}}$.

41. Give yourself the result of question 40, and find the $\frac{3}{4}$ which is in the numerator of the first fraction.

42. Give yourself the result of question 40, and find the $\frac{2}{3}$ which is in the denominator of the second fraction.

43. What fraction divided by the excess of $\frac{3}{7}$ over $\frac{1}{9}$ will give the quotient $2\dfrac{1}{3\frac{1}{4}}$?

44. Find the G. C. M of $\frac{466}{522}$ (nonary), and simplify.

45. Simplify $\frac{45443}{54213}$, both expressed in senary, by finding their G. C. M.

46. Find the L. C. M. of 416 and 620, both in the septenary.

47. If 55143 (senary) is the L. C. M. of two numbers, of which 3113 (senary) is one, and 243 (senary) is the G. C. M., find the other.

48. Multiply $\dfrac{3\frac{4}{6}}{5\frac{}{7\frac{8}{9}}}$ by $\dfrac{1\frac{2}{4}}{3\frac{}{5\frac{6}{7}}}$.

49. Divide $4\dfrac{3}{2\frac{1}{4}}$ by $\dfrac{2\frac{1}{2}-1\frac{1}{4}}{2\frac{1}{2}+1\frac{1}{4}}$.

50. From $3\frac{1}{7}$ take away the excess of $5\frac{1}{5}$ over $4\frac{1}{6}$, and reduce the result to a radix fraction (duodenary).

CHAPTER VI.

Units and Problems requiring their use.

1. Half the difficulty of the majority of problems is to choose the best unit to work in.

2. In the expressions 3 pence a lb., or 10 miles an hour, we have in each case two units, viz. pence and pounds in the first case, and miles and hours in the second.

3. It is of course possible to change these units by reduction; thus 6 miles an hour (where miles and hours are the units) is the same as 6×1760 yards in 60 minutes, or $\frac{6 \times 1760}{60}$, that is, 176 yards a minute, where we have changed our units of miles and hours into yards and minutes.

4. It is not necessary to say that it would be impossible to change a unit, such as yards, into any other such, as pounds or pence.

5. Students for the most part do not have to consider the questions of units until they study such subjects as Hydrostatics and Heat, and yet one unit at least is understood in every question however simple.

6. In problems as to the emptying or filling of casks, the unit usually used is the capacity of the cask, which is certain to generate a great many fractions. By taking a small enough unit we can always work a question without any fractions at all.

To take an example. There are 3 pipes to a cask, one of which (A) can empty the cask in 20 minutes, another (B) in 30 minutes, and another (C) in 40 minutes. If the cask is empty, and whilst it is being filled through taps B and C, A is left open, how long before the cask will be filled? We will solve the problem in two ways side by side. In the first case we will take the capacity of the cask as the unit, and in the second the 120th part of the cask, 120 being the L. C. M. of 20, 30, and 40.

Let the cask be the unit.
$\frac{1}{20}$ of cask flows in through A ea. min.
$\frac{1}{30}$,, B ,,
$\frac{1}{40}$,, C ,,

When B and C are filling the cask and A emptying it $\frac{1}{30}+\frac{1}{40}-\frac{1}{20}$ flows in each minute $=\frac{4+3-6}{120}$

$=\frac{1}{120}$,,

∴ it will take 120 minutes to fill the cask.

Let $\frac{1}{120}$ of the cask be the unit.
6 units flow in through A each min.
4 ,, ,, B ,,
3 ,, ,, C ,,

When B and C are filling it and A emptying it, $4+3-6$ units flow in each minute, or 1 unit; hence, as before, it will take 120 minutes to fill the cask.

7. To take another example—

A cask can be emptied by 4 pipes, A, B, C, and D, in 5 minutes, by A, B, C in 6, by A, C, D in 7 minutes, and by A, B, D in 8 minutes. The cask is full; for two minutes all the taps are open, then A is closed for 1 minute, then B for 1 minute, then C for 2 minutes. How long must A, B, C remain open after this before the cask is empty?

Let us employ as a unit—

$$\frac{1}{5.2.3.7.2.2} \text{ or } \frac{1}{840} \text{ of the cask.}$$

Through A, B, C, D there flow 168 units each minute,
and in through A, B, C ,, 140 ,,
∴ ,, D ,, 28 ,,
again, since through A, C, D ,, 120 ,,
∴ ,, B ,, 48 ,,
again, since through A, B, D ,, 105 ,,
∴ ,, C ,, 63 ,,

∴ through A there flow $168-(28+40+63)$, or 29 units each minute, and 139 units flow through B, C, D. At the end of 6 minutes there have flowed out of the cask

$$336+139+120+210, \text{ or } 805 \text{ units,}$$

there are therefore only 35 units left; ∴ A, B, C must remain open $\frac{35}{140}$, or $\frac{1}{4}$ minute, when the cask will be empty.

8. Questions of the following kind are often considerably simplified by a judicious choice of unit:—

A can do $\frac{3}{5}$ of a piece of work in 7 days, B can do $\frac{6}{7}$ of it in 9 days. If they work together how much more will B have done than A in 10 days?

Let us use as our unit $\frac{1}{5\cdot7\cdot9}$ of the work to be done.

If A can do $\frac{3}{5}$ of the work in 7 days, he of course does $\frac{3}{5\cdot7}$ in each day, which contains $\frac{1}{5\cdot7\cdot9}$ 27 times.

$\therefore$ A can do 27 units each day,
and B „ 30 „
$\therefore$ at the end of 10 days A will have done 270 units,
and „ B „ 300 units,

or B will have done 30 units, or $\frac{30}{5\cdot7\cdot9}=\frac{2}{21}$ of the work to be done.

I of course adopt the unit $\frac{1}{5\cdot7\cdot9}$ because I want a number exactly contained in $\frac{1}{5}$, because A can do $\frac{3}{5}$ of the work in a given time; again, it must be contained in $\frac{1}{7}$, because of the 7 days; and lastly, it must be contained in $\frac{1}{9}$, because of the 9 days.

9. If A can do as much work in 5 hours as B can do in 6 hours, or as C can do in 9 hours, how long will it take C to complete a piece of work one half of which has been done by A working twelve hours and B working 24 hours?

Let us adopt as a unit the $\frac{1}{5.2.3.3}$ of the amount of work A can do in 5 hours, or B in 6 hours, or C in 9 hours,

then A can do 18 of these units each hour,
„ B „ 15 „
„ C „ 10 „
In 12 hours A has done 216 (12×18) units,
„ 24 „ B „ 360 (24×15) „

and $216+360=576$, which is half the work; C therefore has 576 units to do, which he accomplishes in $\frac{576}{10}$, or $57\frac{3}{5}$ hours.
Note.—We do not multiply out the 5.2.3.3, as we never require the number.

10. A piece of work has to be finished in 36 days, and 15 men are set to do it, working 9 hours a day; but after 24 days it is found that only $\frac{3}{5}$ of the work is done. If 3 additional men be then put on, how many hours a day will they all have to labour to finish the work in time?

Our best unit here will evidently be an hour's work of each man.

Then in 24 days $15 \times 9 \times 24$ units are done, but this is only $\frac{3}{5}$ of the work;

$$\therefore\ 5 \times 9 \times 24 \text{ units is one-fifth,}$$

$$\text{and } 25 \times 9 \times 24 \text{ units is the work to be done.}$$

Now $\frac{2}{5} = 2 \times 5 \times 9 \times 24$, and this is done by 18 men in 12 days, and 18×12 is contained in $2 \times 5 \times 9 \times 24$ ten times, which shows how many hours a day each must have worked.

11. To take another example—

A garrison of 1500 men is provisioned for 50 days, but after 25 days 200 women and 700 children are brought into the garrison, and it is found that 8 men eat as much as 11 women, or as 14 children. How long will the provisions hold out?

Here we must take a unit which will enable us to express without fractions the different amount of provisions that the man, woman, and child severally eat each day.

Let us adopt as a unit $\dfrac{1}{2.4.11.7}$ of what each man eats in 8 days, or each woman eats in 11 days, or each child eats in 14 days;

$$\begin{aligned} \therefore \text{ each man eats } & 77 \text{ such units each day,} \\ \text{woman } ,, \ & 56 \quad ,, \\ \text{child } ,, \ & 44 \quad ,, \end{aligned}$$

The original provisions then consisted of $1500 \times 50 \times 77$ such units.

In the 25 days before women and childen came, the men had consumed $25 \times 1500 \times 77$, and there were $25 \times 1500 \times 77$ units left; but each day after the women and children had arrived, there were consumed $1500 \times 77 + 200 \times 56 + 700 \times 44$

and this number is contained in $25 \times 1500 \times 77$ this number of times, viz. $\dfrac{25 \times 1500 \times 77}{1500 \times 77 + 200 \times 56 + 700 \times 44}$

$$= \frac{25 \times 15 \times 11}{15 \times 11 + 2 \times 8 + 4} = \frac{44125}{165 + 16 + 44}$$

$$= \frac{4125}{225} = \frac{165}{9} = 18\tfrac{1}{3},$$

which gives us the number of days, viz. $18\frac{1}{3}$.

12. A alone can do a piece of work in 15 days, B alone in 20 days, and C alone in 25 days. In what time would they do it if they all worked together?

Let us adopt as our unit $\dfrac{1}{3.5.2.2.5}$ of the entire work to be done;

then A can do 20 of these units each day,
B „ 15 „
C „ 12 „

that is 47 units each day, and there are 300 units to perform;

$\therefore$ it would take A, B, C $\frac{300}{47} = 6\frac{18}{47}$ days.

13. Here is an example taken from a Civil Service paper.

If 12 men, 10 women, and 22 children working together earn £16, 13s. in a week of 6 days, and one woman earns as much as 3 children, and a man and a child as much as 3 women, what are the daily wages of each man, woman, and child?

Let a child's daily wage be one unit.

Then a woman earns each day 3 units,
and a man „ $(9-1)$ 8 units;
$\therefore$ in 6 days a child earns 6 units,
„ woman „ 18 „
„ man „ 48 „

$\therefore$ in 6 days 12 men, 10 women, and 22 children earn 576 + 180 + 132 units, which we know to be 333×12 pence ;

$$\therefore \text{ each unit is } \frac{\overset{3}{\cancel{333}} \times \overset{3}{\cancel{12}}}{\underset{\underset{2}{\cancel{8}}}{\cancel{888}}} \text{ pence} = \frac{9}{2} \text{ pence.}$$

A child's daily wage is $4\frac{1}{2}$d., a woman's 1s. $1\frac{1}{2}$d., and a man's 3s.

14. Here is one more example, also taken from a Civil Service paper.

If a person spend £50 more than $\frac{19}{20}$ of a fixed income in a certain year, £35 less than $\frac{14}{15}$ of it in the next year, and $\frac{11}{12}$ of it in the year after that, and his saving in the three years amounts to £705, what is his income ?

To save fractions in the working we will take as our unit $\frac{1}{3.5.2.2}$ of the income ;

$$\begin{array}{rcl} \text{then } \frac{19}{20} \text{ of it contains} & & 19 \times 3 \text{ units,} \\ \frac{14}{15} \quad ,, & & 14 \times 4 \text{ units,} \\ \text{and } \frac{11}{12} \quad ,, & & 11 \times 5 \text{ units.} \end{array}$$

$$\begin{array}{lll} \text{In the 1st year he spends} & 19 \times 3 \text{ units} & + £50, \\ \quad ,, \quad \text{2nd} \quad ,, & 14 \times 4 \quad ,, & - £35, \\ \quad ,, \quad \text{3rd} \quad ,, & 11 \times 5 \quad ,, & \end{array}$$

$\therefore$ altogether out of 3×60 units he spends $57 + 56 + 55$ and £15 ;

$\therefore$ his savings are 12 units $-$ £15, but these are equal to £705 ;

$\therefore$ each unit is $\frac{720}{12}$, or £60, or his annual income is £60×60 = £3600.

This subject of units will be further discussed in future chapters.

EXAMINATION AND EXAMPLES.

1. In the expression 50 miles an hour, what are the units used? Change them into feet and a second.

2. In the expression 15 horses for 2 days of 12 hours cost £2,—change the units, horse, £, day, into donkey, shillings, hours, when the working powers of a horse is double that of a donkey.

3. If 3 men earn £10 in 8 weeks (6 days of 12 hours). Change the units into boys, pence, minutes, a boy earning a quarter what a man does.

4. If 3 men *or* 5 women can finish a piece of work in 12 hours, in what time would 2 men *and* 6 women do it?

5. 6 men eat as much as 8 women, or 10 children. In how many days would 100 women and 100 children eat the provisions collected for as many men for 300 days?

6. A and B can together perform a certain work in 30 days; at the end of 15 days, however, B is called off and A finishes it alone in 20 days more. Find the time in which each could perform the work alone.

7. A, B, and C can drink a cask of beer in 15 days. A and B together drink four-thirds of what C does, and C drinks twice as much A. Find how long A would be in drinking the cask.

8. If the answer to the last question is 70 days, prove that C's daily consumption is double that of A's.

9. Given the 70 days and all the other data but the 15 days, find it.

10. Given the 70 days and all the data but the fact that three times what A and B drink is equivalent to four times what C drinks, and find this relationship.

11. Divide the number 60 into two parts, such that a seventh of one part is equal to an eighth of the other.

12. A man purchases some horses, oxen, sheep, pigs, and fowls; for every horse he pays the same price as he pays for 2 oxen, 12 sheep, 10 pigs, and 100 fowls. If each ox is worth £8, 15s., what would he pay for 3 horses, 7 oxen, 100 sheep, 40 pigs, and 50 fowls?

13. With the same data as the last question, except that instead of an ox being worth £8, 15s. a pig is worth £6, 7s. 6d., what would be the value of the same animals?

14. If 6 men earn £54 whilst 10 women earn £40 and 12 children earn £12, how long would they take altogether to earn £848, supposing a child earns 1s. 8d. a day?

15. Give yourself the answer to the last question, and also the data except one element, and find it.

16. Do this with all the elements found in the problem, and write down the questions.

17. How many hours a day must 24 men work to accomplish as much in 5 days as 25 men could do in 4 days if they worked six hours a day? Take the work per hour for each man as your unit.

18. Give yourself the answer of the last question and all the other data, and find them one after the other, writing out the questions as they would be set.

19. Of the boys in a school one-third are over 15 years of age, one-third between 15 and 10. Divide £100 amongst them, giving 10s. to each boy over 15, 6s. 8d. to each between 10 and 15, and 3s. 4d. to the rest. How many boys are there? Take as your unit the amount of money given to the boys under 10.

20. Of the boys in a school $\frac{1}{2}$ are over 15, $\frac{1}{3}$ are between 15 and 10. Divide £660 amongst them, so that each boy over 15 has five times as much as those under 10, and those between 15 and 10 have three times as much. What do the boys over 15 receive?

21. If those between 15 and 10 receive £5 each, how many boys are there in the school?

22. If there were 324 boys in the school, how much would each boy over 15 receive?

23. The sum of £180 is divided amongst 20 men, 40 women, and 80 children so that a man and a child have double a woman's share. If the women have £60, what will each child receive?

24. A man undertook to build a wall in 40 days, and engaged 25 men to do it. After 16 days he engaged 15 more, how soon do they finish it?

25. A man engaged to build a wall in 40 days, and employed a certain number of men; but after 16 days he employed 15 more, and then finished the work in 15 days. How many did he originally employ?

26. A man employed 25 men to build a wall in a certain number of days, but wishing to hurry it, after 16 days he en-

gaged 15 more, and then the work was finished in 15 days. In how many days would the original men have finished it?

27. If it cost as much to feed 3 men as to feed 4 boys, and for 3 boys the cost is found to be 19s. $2\frac{1}{4}$d. per week, how much per week will it be for 51 men? Take as your unit $\frac{1}{3 \times 4}$ of what it costs to feed 3 men or 4 boys a week.

28. Take the answer to last question and find the 51 men.

29. Eleven cubic inches of iron weigh as much as 7 cubic in. of lead, and the price per ton of lead is £15, of iron £4. The value of a certain block of lead is £36, 17s. 11d. What would be the value of a block of iron of the same size?

Take as your unit of weight the $\frac{1}{11 \times 7}$ of the weight of 11 cubic in. of iron, or 7 cubic in. of lead.

30. A chest containing 350 oranges is bought at Naples for 42 pence, the cost of carriage is $\frac{1}{10}$ the original cost. The oranges are retailed at rate of 10 for 3d. Find the profit on 100 oranges.

31. Give yourself the answer of 30 and all the data except the 350 oranges, and find it.

32. Give yourself the answer of 30 and all the data except the selling price, and find it.

33. Give yourself the answer of 30 and all the data except the rate of carriage, and find it.

34. The inhabited house duty is $\frac{3}{80}$ of the rent, and the income tax is $\frac{1}{40}$ of the rent. If the difference between these two amounts is £3, 10s., find the rent.

Take as your unit $\frac{1}{8 \times 10}$ of the rent.

35. If the inhabited house duty is £3, 10s. more than a tax of $\frac{1}{40}$ of the rent, which is £280, what fraction of the rent is the house duty?

36. A sum of £3070, 3s. 3d. is divided amongst A, B, and C, A has $\frac{1}{3}$ of $\frac{4}{9}$ of B's share, and A and C together $\frac{5}{4}$ of B's. Find their shares.

37. Give yourself A's share (£202, 3s. $0\frac{4}{27}$d.) and all the other data but the sum of money, and find it.

38. Give yourself B's share and all the other data except the connection between A's and B's money, and find it.

39. Give yourself C's money and all the data except the connection between B's and C's money, and find it.

40. A creditor receives $\frac{37}{80}$ of his debt, and afterwards $\frac{3}{16}$ of the remainder. If his debt were £592, what did he receive?

Take as your unit $\frac{1}{4.4.4.15}$ of the debt.

41. If 5 horses eat as much as 76 sheep, and 20 horses and 196 sheep cost £7, 15s. to keep 9 days, what sum will keep 15 horses and 72 sheep for 8 days?

42. Give yourself the answer and all the data except the 72 sheep, and find it.

43. Give yourself the answer and all the data except the number of horses that can be kept for the same price as 76 sheep.

44. A woman after spending $\frac{2}{5}$ of her money finds that $\frac{4}{7}$ of what remains is 1s. 9d., how much had she?

Take as your unit $\frac{1}{5.7}$ of her money.

45. Give yourself the answer of 44, and find the fraction of money the woman spent at first.

46. Three men can do as much work as 5 boys, and the wages of 3 boys are equal to those of 2 men. A work on which 40 boys and 15 men are employed takes 8 days, and costs £350. How long would it take if 20 boys and 20 men were employed?

47. How much would it cost, the data being the same as in 46?

48. Give yourself the cost and the other data except the relationship between the men and the boys' wages, and find it.

49. A walks to a place at the rate of $4\frac{1}{2}$ miles an hour; at 8 miles from his destination he meets B and turns back with him (walking at B's pace) for a mile. If A is half an hour late at his destination, what is B's rate?

50. With the data of 49 find also at what rate A should have walked after parting with B so as to arrive at the proper time.

CHAPTER VII.

Evaluating Fractions (both Vulgar and Decimal) of Concrete Quantities, and expressing Quantities as Fractions (both Vulgar and Decimal) of other Quantities.

1. By the definition of a fraction (say $\frac{1}{5}$), to obtain its value—in this case the fifth of anything—we must divide it into five equal parts, and take one of them. Thus, $\frac{1}{5}$ of £41, 7s. $2\frac{1}{4}$d. is found thus—

$$\begin{array}{r@{\;}rrr} & \pounds & s. & d. \\ 5) & 41 & 7 & 2\frac{1}{4} \\ \hline & 8 & 5 & 5\frac{1}{4}, \\ \hline \end{array}$$

and $\frac{3}{5}$ of £41, 7s. $2\frac{1}{4}$d. will be three times £8, 5s. $5\frac{1}{4}$d., or £24, 16s. $3\frac{3}{4}$d.

2. If we have to divide a quantity by 5 and then multiply the result by 3, it is immaterial which of these operations we perform first.

Since multiplication can never generate a fraction where there was none before, it is generally better to perform the multiplication first, though the numbers in the intermediate line will be greater than they would be if the division were performed first. To work a question both ways side by side—

Find $\frac{3}{5}$ of £2, 6s. 1d.

$$\begin{array}{r@{\;}rrr} & \pounds & s. & d. \\ & 2 & 6 & 1 \\ & & & 3 \\ \hline 5) & 6 & 18 & 3 \\ \hline & 1 & 7 & 7\frac{4}{5} \\ \hline \end{array} \qquad\Bigg|\qquad \begin{array}{r@{\;}rrr} & \pounds & s. & d. \\ 5) & 2 & 6 & 1 \\ \hline & & 9 & 2\frac{3}{5} \\ & & & 3 \\ \hline & 1 & 7 & 7\frac{4}{5} \\ \hline \end{array}$$

On the left-hand side we have no fractions whatever until the last line; but on the right we have only 9s. $2\frac{3}{5}$d. to deal with, instead of the larger quantity, £6, 18s. 3d.

In these questions, as in Practice, unless you are particularly asked to take the working out to the fraction of a farthing, it is better to only carry it to the fraction of a penny, since we *write* farthings as a fraction of a penny.

Supposing you are working examples in which the answers are given in farthings and fractions of a farthing, and you may wish to express them in pence and fractions of a penny, to test the accuracy of your working, divide the fraction of a farthing by 4 and add it to the farthings, and the result will be the fraction of a penny.

Thus, supposing the answer were $4\frac{3}{4}\,\frac{10}{11}$d., to reduce the $\frac{3}{4}$ and $\frac{10}{11}$ of a farthing to pence we must divide $\frac{10}{11}$ by 4, which gives us $\frac{10}{44}$, and add this on to $\frac{3}{4}$, which gives us $\frac{33+10}{44}=\frac{43}{44}$d.

3. Since division is the exact converse of multiplication, to divide a concrete quantity by a fraction—say $2\frac{1}{3}$ or $\frac{7}{3}$, we practically multiply it by $\frac{3}{7}$, or find $\frac{3}{7}$ of it.

E.g. divide 5 po. 2 yds. 1 ft. by $2\frac{1}{3}$.

$$
\begin{array}{rrrrr}
 & \text{po.} & \text{yds.} & \text{ft.} & \\
 & 5 & 2 & 1 & \\
 & & & 3 & \\
\hline
7) & 16 & 1\frac{1}{2} & 0 & \\
\hline
 & 2 & 1 & 2 & 4\frac{2}{7}
\end{array}
$$

Now what is the meaning of this result? It means that if I were to take this quantity $2\frac{1}{3}$ of a time, I should obtain 5 po. 2 yds. 1 ft.

4. If a single fraction of a denomination has to be reduced to another, it is generally better to perform the multiplication or division by signs. Thus, express £$\frac{3}{40}$, $\frac{96}{97}$d. as shillings.

$$£\frac{3}{40}=\frac{3\times 20}{40}\text{s.}=\frac{3}{2}\text{s.}$$

$$\frac{96}{97}\text{d.}=\frac{96}{97\times 12}\text{s.}=\frac{8}{97}\text{s.}$$

5. By means of fractions, any quantity, however small, can

be expressed in as high a denomination as we like. *E.g.* express $\frac{3}{4}$d. in pounds—

$$\tfrac{3}{4}\text{d.} = \frac{3}{4.12}, \text{ or } \frac{1}{16}\text{s.} = \frac{1}{16.20}, \text{ or } £\frac{1}{320}.$$

6. Express in shillings the following quantities, and add them together:—$\frac{1}{4}$d.; 2s. 3d.; £1, 6s. 6d.; £$\frac{5}{7}$; $\frac{2}{3}$ of £100.

$$\left.\begin{array}{rlrr}
\tfrac{1}{4}\text{d.} = & & \tfrac{1}{48} \text{ shil.} & 7 \\
\text{2s. 3d.} = & & 2\tfrac{1}{4} \quad ,, & 84 \\
£1, \text{ 6s. 6d.} = & & 26\tfrac{1}{2} \quad ,, & 168 \\
£\dfrac{5}{7} = \dfrac{5 \times 20}{7} = & & 14\tfrac{2}{7} \quad ,, & 96 \\
\tfrac{2}{3} \text{ of } £100 = \dfrac{2 \times 100 \times 20}{3} = & & 1333\tfrac{1}{3} \quad ,, & 112 \\
\hline
 & & 1376\tfrac{131}{336} & 467
\end{array}\right\} 2.2.2.2.3.7.$$

7. To evaluate decimals of concrete quantities, reduce them as in whole numbers.

E.g. find the value of £4·859375.

$$\begin{array}{l}
4·859375 = £4, \text{ 17s. } 2\tfrac{1}{4}\text{d.} \\
\quad\quad 20 \\
\hline
17·1875\cancel{0}\cancel{0} \text{ shillings} \\
\quad\quad 12 \\
\hline
2·25\cancel{0}\cancel{0} \text{ pence} \\
\quad 4 \\
\hline
1·00 \text{ farthing.}
\end{array}$$

As we want to find what number of whole shillings, pence, etc., there are, we do not change them into farthings.

To express £4, 17s. $2\frac{1}{4}$d. as pounds decimally, will be, of course, the exact converse of this. Begin with the farthings,

and reduce them to pence. Place the whole pence in the proper place, and reduce them to shillings, etc.

$$\begin{array}{r|l} 4) & 1\cdot \text{ farthing} \\ 12) & 2\cdot2500 \text{ pence} \\ 20) & 17\cdot1875 \text{ shillings} \\ & £4\cdot859375. \end{array}$$

8. If we are to express a quantity as £2, 4s. 8¼d. in terms of £5, 10s., the operation may be performed in the same way. Thus, reduce £2, 4s. 8¼d. to the decimal of £5, 10s.

$$\begin{array}{r|l} 4) & 1\cdot \text{ farthing} \\ 12) & 8\cdot25 \text{ pence} \\ 20) & 4\cdot6875 \text{ shillings} \\ & 2\cdot234375 \text{ pounds} \\ & 2 \\ 5\tfrac{1}{2} = \tfrac{11}{2} \left\{ 11) \right. & 4\cdot46875\text{ half pounds} \\ & \cdot40625. \text{ Five pounds ten shillings.} \end{array}$$

To make this decimal divide exactly, we had to take a quantity which was both divisible by 3, on account of the 3 in the 12,—12d. = 1s.,—and by 11, on account of the 11 in £5, 10s.; but this will be explained fully in chapter on Recurring Decimals.

9. To express one quantity as a fraction of another quantity of the same kind; *e.g.* reduce 2s. 3d. to the fraction of 3s. 4d. Since 3s. 4d. = 40d., and each penny is (by definition) $\frac{1}{40}$ of this, 2s. 3d. or 27 pence is $\frac{27}{40}$ of 3s. 4d. This operation is really a comparison as to how many times the one quantity is contained in the other. To take another example: What part of 3 po. 2 yds. 1 ft. is 1 po. 1 yd. 2 ft.? The answer is $\frac{\text{1 po. 1 yd. 2 ft.}}{\text{3 po. 2 yds. 1 ft.}}$; or, reducing and (1) cancelling the de-

nominator feet, and (2) multiplying numerator and denominator by 2, we find 1 po. 1 yd. 2 ft. is $\frac{43}{113}$ of 3 po. 2 yds. 1 ft.

10. Nothing is more important than to notice that the answers to these questions are abstract, and hence that it is possible to multiply a concrete quantity by them.

E.g. multiply £4, 7s. 6d. by the fraction that 3 lbs. 2 oz. is of 4 lbs. 1 oz. 10 dwts.

$$3 \text{ lbs. } 2 \text{ oz. is } \frac{38}{49\frac{1}{2}} \text{ or } \frac{76}{99} \text{ of } 4 \text{ lbs. } 1 \text{ oz. } 10 \text{ dwts.}$$

$$\begin{array}{rrrr} & \pounds & s. & d. \\ & 4 & 7 & 6 \\ & & & 76 \\ \hline 9) & 332 & 10 & 0 \\ \hline 11) & 36 & 18 & 10\frac{2}{3} \\ \hline & 3 & 7 & 2\frac{2}{33} \\ \hline \end{array} \qquad \begin{array}{rl} 2)76 & \text{sixpences} \\ \hline 38 & \text{shillings} \\ 522 & \\ \hline 20)570 & \text{shillings} \\ \hline \pounds 28 & 10 \\ 304 & 0 \\ \hline \pounds 332, & 10s. \\ \hline \end{array}$$

11. It is often very convenient to express a compound expression, as £2, 4s. 8½d., as pounds. Thus—

$$\pounds 2\frac{4\frac{8\frac{1}{2}}{12}}{20}, \text{ which reduces thus—} \pounds 2\frac{4\frac{17}{24}}{20} = \pounds 2\frac{113}{480} = \pounds\frac{1073}{480}.$$

Or we might adopt the same method as we do in reducing quantities to the decimal of a higher denomination. Thus—

$$\begin{array}{l} 12)8\frac{1}{2} \text{ pence} \\ \hline 20)4\frac{17}{24} \text{ shillings} \\ \hline \pounds 2\frac{113}{480} \text{ or } \pounds\frac{1073}{480}, \\ \hline \end{array}$$

where, after we have expressed the lower denomination as the

next higher, we add on the whole number of that higher denomination.

12. To change a vulgar fraction into a decimal, all that we have to do is to perform the division implied in the fraction. Thus, $\frac{3}{4}$ means $3 \div 4$. If we divide 3 by 4, we must first fix the 3 as units, which we do by affixing a decimal or unit point, and then add on as many ciphers as are necessary.

$$\begin{array}{r} 4\underline{)3{\cdot}00} \\ \underline{\underline{0{\cdot}75}} \end{array}$$

Of which this is the explanation. 4 units into 3 units will not go, so we put down 0, and mark them as units, and reduce the 3 units to 30 tenths, which we divide by 4, and get 7 tenths, and 2 over. These 2 tenths we convert into 20 hundredths, which will give us a quotient 5 hundredths; hence $\frac{3}{4} = 0$ units 7 tenths and 5 hundredths, or $\cdot 75$.

Since the denominator of a decimal fraction is 10, 100, or some *power* of ten (that is, the product of some number of tens multiplied together), and we can only change by multiplication tens and fives into tens and their powers, only those fractions (in their lowest terms) whose denominators consist of nothing but tens and fives can be converted into exact decimals, called 'terminating' decimals. $\frac{3}{64}$, $\frac{17}{125}$, $\frac{7}{512}$ can all be converted into terminating decimals, and $\frac{57}{192}$ can also, because though there is a factor 3 in the denominator, there is one that cancels it in the numerator; but $\frac{2}{3}$, $\frac{5}{7}$, $\frac{4}{11}$ cannot.

13. To change a quantity into the decimal of a compound quantity, we must of course reduce them to the same denomination, and divide the quantity by the quantity to which we are to reduce it to the decimal.

E.g. reduce £3, 2s. $2\frac{1}{2}$d. to the decimal o £8, 10s. 8d. £3, 2s. $2\frac{1}{2}$d. = 1493 halfpence and £8, 10s. 8d. = 4096 halfpence.

```
4096)1493·0(0·364501953125
     1228 8
     ------
      264 20
      245 76
      ------
       18 440
       16 384
       ------
        2 0560
        2 0480
        ------
           8000
           4096
           ----
           39040
           36864
           -----
            21760
            20480
            -----
             12800
             12288
             -----
               5120
               4096
               ----
               10240
                8192
               -----
                20480
                20480
                -----
                  . .
```

∴ £3, 2s. $2\frac{1}{2}$d. = ·364501953125, or $\frac{364501953125}{1000000000000}$ of £8, 10s. 8d., and of course the fraction $\frac{364501953125}{1000000000000} = \frac{1493}{4096}$.

14. By the definition of a decimal as a whole number carried beyond unity, $\cdot57 = \frac{57}{100}$, or there are as many ciphers after the 1 of the denominator of the equivalent vulgar fraction as there are places in the decimal.

$$\cdot006 = \tfrac{6}{1000} \text{ or } \tfrac{3}{500},$$
$$1\cdot0017 = 1\tfrac{17}{10000} \text{ or } \tfrac{10017}{10000}.$$

15. If decimals appear in the numerator or denominator or both of a vulgar fraction, it is far better to multiply both numerator and denominator by 1, followed by as many ciphers as there are figures in the decimal with the most decimal places, and so reduce them to whole numbers.

$$\text{Thus: } \frac{3\cdot25}{4\cdot17} = \frac{325}{417}, \text{ or } \frac{45\cdot3}{1\cdot23} = \frac{4530}{123}.$$

As there are two decimal places in the denominator of the second fraction, we must multiply both numerator and denominator by 100, or the decimal point must be moved two places to the right. To work a difficult example—

$$3\cdot57\frac{4\frac{7\cdot3}{2\cdot5}}{3\frac{1\cdot25}{22\cdot5}} = 3\frac{57}{100} + \frac{4\frac{73}{25}}{3\frac{125}{2250}}$$

$$= 3\frac{57}{100} + \frac{173}{75 + \frac{125}{90}}$$

$$= 3\frac{57}{100} + \frac{15570}{6750 + 125} = 3\frac{57}{100} + \frac{\overset{3114}{\cancel{15570}}}{\underset{1375}{\cancel{6875}}}$$

$$= 3\frac{57}{100} + 2\frac{364}{1375} = 5\frac{8835 + 1456}{5.5.2.2.5.11}$$

$$= 5\frac{10291}{5500} = 6\frac{4791}{5500}.$$

16. In reducing a quantity to the decimal of a compound quantity, we may often save ourselves time by reducing them both, not to the lowest denomination that appears in the question, but to some higher one. Thus: What decimal of £2, 0s. 3d. is £1, 2s. $7\frac{1}{2}$d.? Now £2, 0s. 3d. = 40·25 shillings, and £1, 2s. $7\frac{1}{2}$d. = 22·6 shillings; if, therefore, we divide 22·6 by 40·25, we shall get the decimal required.

EXAMINATION AND EXAMPLES.

1. Subtract $\frac{15}{16}$ of £16, 2s. 4d. from ·0125 of £1626, 15s., and find by what decimal the result must be multiplied to produce £1, 6s. $1\frac{1}{2}$d.

2. Add together $\frac{5}{7}$ of a guinea, $\frac{3}{8}$ of a sovereign, $\frac{7}{20}$ of a crown, and $\frac{1}{4}$ of a shilling. Express the result as the fraction of $4\frac{7}{20}$ of £1.

3. Add together $3\frac{5}{8}$ pounds, $9\frac{3}{7}$ guineas, and $2\frac{4}{5}$ half-crowns.

4. Find $\dfrac{\frac{1}{2}+\frac{1}{3}-\frac{1}{6}}{\frac{1}{2}\text{ of }\frac{1}{3}\text{ of }\frac{1}{6}}$ of £3, 4s.

5. Simplify $\dfrac{3\frac{1}{7}-\frac{2}{3}}{\frac{4}{49}\times 7\frac{7}{12}}$; and divide ·00036 by ·006.

6. Divide 24·109932 by 301·28; and find the value of ·90625 of a cubic yard.

7. Find the value of $\frac{2}{7}$ of $\frac{21}{22}$ of £15, 10s. 9d. − $\frac{7}{130}$ of $3\frac{1}{3}$ of $2\frac{11}{14}$ of £2, 15s. 8d., and express the result as the fraction of half-a-crown.

8. Add together $\frac{6}{11}$, $6\frac{1}{11}$, and $\frac{6}{11}$ of $6\frac{1}{11}$; and express 3 quarts 1 pint as the fraction of 3 bushels 1 peck.

9. If $\frac{7}{15}$ of a guinea be taken from $\frac{3}{12}$ of $\frac{5}{9}$ of £5, what fraction of £3, 9s. will remain?

10. Find the value of $\frac{384}{990}$ of £8, 16s. 3d. + $6\frac{1}{2}$ of $\frac{16}{65}$ of 7s. $8\frac{1}{2}$d. + $\frac{7}{11}$ of 1d.

11. How many sovereigns are equivalent to 8340 guineas?

12. Simplify the fraction—

$$\frac{(\frac{1}{2}+\frac{2}{3}) \text{ of } (\frac{3}{4}+\frac{4}{5})+(\frac{1}{2}+\frac{3}{4}) \text{ of } (\frac{2}{3}+\frac{4}{5})+(\frac{1}{2}+\frac{4}{5}) \text{ of } (\frac{2}{3}+\frac{3}{4})}{\frac{1}{2} \text{ of } (\frac{2}{3}+\frac{3}{4}+\frac{4}{5})+(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}) \text{ of } \frac{4}{5}}.$$

13. Find the value of $\frac{342}{990}$ of 9s. 2d. + $\frac{5}{6}$ of ·075 of £10 + ·05 of 1·125 of £1, 13s. 4d., and express the result as a decimal fraction of £50.

14. Find the value of $\frac{3}{8}$ of £1, 10s. + $\frac{2}{3}$ of $1\frac{1}{2}$ of $\frac{4}{5}$ of £1, 5s. − ·5625 of £1; and reduce 2 cwt. 3 qrs. 3 lbs. 8 oz. to the decimal of a ton.

15. Find the value of $\frac{1}{4}$ of 17s. 8d. + 2·625 of 1s. − $\frac{3}{4}$ of $\frac{5}{8}$ of 5s. 4d. + $\frac{237}{900}$ of 25s., and reduce the result to the decimal of £5.

16. Find the amount of land in $\frac{4}{5}$ ac. + $\frac{1}{36}$ ro. + $\frac{8}{9}$ po.; and the difference between £$\frac{15}{36}$ + 56s. and £·416 + ·83s.

17. Simplify $\dfrac{\frac{1}{3}(\frac{1}{2}+\frac{1}{7})-\frac{1}{7}}{1+\dfrac{1}{2-\frac{1}{4}}}$.

18. Compare the values of the fractions—

$$\frac{11 \times 4}{5 \times 9}, \frac{12 \times 3}{4 \times 10}, \frac{10 \times 5}{6 \times 8}, \text{ and } \frac{11+4}{5+9},$$

placing them in their order of magnitude.

19. Find the value of $\frac{3}{16}$ florin + $\frac{3}{5}$ of 2s. 6d. subtracted from ($\frac{1}{10}$ of 5s. + £$\frac{3}{4}$), and reduce the result to the decimal of £5.

20. What fraction of $9\frac{1}{4}$ guineas is $\frac{7}{16}$ of £9, 17s. 4d.?

21. When eggs are at 24 a shilling, how many must be given in payment of a debt of £1, 11s. $5\frac{1}{2}$d.?

22. Express 2 ft. $7\frac{1}{2}$ in. as the decimal fraction of 100 yds.

23. A dairyman buys milk at $2\frac{1}{2}$d. per quart, dilutes it with water, and sells the mixture at 3d. per quart; his profits are 60 per cent. upon his outlay. How much water does he mix with each quart of milk?

24. Add together $\frac{2}{9}$ of 1 guinea, $\frac{1}{7}$ of £1, 6s. 4d., and $\frac{1}{5}$ of 3s. 8d., and express their sum as a fraction of 6s. 8d.

25. Reduce $2\frac{1}{2}$ gills to the vulgar and decimal fractions of $3\frac{1}{3}$ gallons.

26. Add together $\frac{2}{3}$ of a crown, $\frac{13}{126}$ of a guinea, $\frac{1}{3}$ of 18s. 6d., and $\frac{15}{36}$ of £1.

27. A man buys 500 quarters of wheat at 56s. a quarter. He sells one-half of this quantity at the rate of 6s. a bushel. At what rate must he sell the remainder so as to gain £25 by the whole transaction?

28. What is the value of 3·1475 if the unit is £5, 2s. 6d.?

29. Add together $\frac{3}{5}$ of £1, 6s. $6\frac{3}{4}$d., $\frac{11}{336}$ of a guinea, $\frac{5}{64}$ of a sovereign, ·4375 of a shilling, and $\frac{2}{15}$ of half-a-sovereign.

30. Reduce $2\frac{1}{8}$ of £5, 11s. to the fraction of $2\frac{5}{16}$ of £4, 5s.

31. Six coins of equal weight, made of gold and silver mixed, are melted together and recast. In one of them the gold is $\frac{2}{3}$ of the silver, in two others it is $\frac{3}{8}$, and in the rest it is $\frac{7}{12}$. What fraction will the gold be of the silver in the new coins?

32. Express the difference between £$7\frac{3}{7}$ and $\frac{3}{7}$ of £7 as a fraction of £7·3.

33. Two chronometers, which are precisely together on a certain day at noon, are at the same instant on the next day but one 2·4 seconds apart. Supposing the slower one to be correct, find the true value of a second as shown by the faster.

34. Express 89 galls. 1 qt. 1 pt. as the decimal of 572 galls.

35. Divide $\frac{7}{16}$ of $\frac{9}{11}$ by $2\frac{1}{3}$ of $1\frac{1}{4}$. How many square yards are there in the fraction of an acre the result represents?

36. Convert $\frac{1}{64}$ and $\frac{22}{3125}$ into decimal fractions; divide the second result by the first, and convert the quotient into a vulgar fraction in its lowest terms.

37. If a litre is ·22 gallons, find to the nearest penny in English money the value of a pint of liquid which is worth 10 francs the litre, 1200 francs being equivalent to £49.

38. What is the least weight which can be expressed either by a number of troy pennyweights or by a number of avoirdupois ounces? Give the answer in troy weight.

39. Reduce nine inches and nine-tenths to the decimal of a mile.

40. Find the value of—

$$\frac{15\frac{3}{5}}{7\frac{4}{5}} \text{ of } £1 + \tfrac{1}{5} \text{ of } £140,\ 10\text{s. } 6\text{d.} + \tfrac{3}{5} \text{ of a guinea.}$$

41. On stream B is intermediate to and equidistant from A and C. A boat can go from A to B and back in 5 hrs. 15 min., from A to C in 7 hrs. How long would it take to go from C to A?

42. I have a certain sum of money wherewith to buy a certain number of nuts, and I find that if I buy at the rate of 40 a penny I shall spend 5d. too much, if 50 a penny, 10d. too little. How much have I to spend?

43. Find the value of 500 times the difference between $\frac{1}{84}$ of $2\frac{1}{2}$ cwt. and $\frac{1}{80}$ of 1 cwt. 3 lbs.

44. Divide £147 by 5·137.

45. A gives away in charity $\frac{1}{8}$ of his income, and pays $\frac{1}{10}$ of it in rates and taxes. With these deductions he has £473, 13s. 1d. left. What is his gross income?

46. Divide £29 into an equal number of half-sovereigns, crowns, half-crowns, shillings, sixpences, and fourpences.

47. Find $\frac{5}{7}$ of £3, 4s. $3\frac{3}{4}$d. + ·1875 of £5, 11s. 8d. + $\frac{2}{3}$ of $\frac{9}{11}$ of 4s. $9\frac{3}{4}$d. + ·001 of £1, 0s. 10d., and reduce it to decimal £5.

48. A gallon contains 277·274 cubic in.; a cubic foot of water weighs 1000 oz. How many gallons will weigh a ton? and what is the weight of a pint?

49. Evaluate 8·71875 of 8d. + 1·46875 of 6s. 8d. − ·0625 of 8 guineas.

50. Evaluate $\frac{3}{4}$ of $\frac{1}{9\frac{1}{2}}$ of £1, 18s. + $\frac{2}{3}$ of ·375 of 15s. + $\frac{2}{3}$ of $\frac{425}{990}$ of 8s. 3d.

CHAPTER VIII.

Reciprocals and their Applications.

1. If A have 3s., each of his shillings is one-third of his money; and if B have 5s., each of his shillings is one-fifth of his money.

Now A has 3 of B's fifths, or in other words A has $\frac{3}{5}$ of B's money, but B has 5 of A's thirds, therefore B has $\frac{5}{3}$ of A's money.

$\frac{5}{3}$ is called on this account the reciprocal of $\frac{3}{5}$.

2. Since $\frac{5}{3} = \frac{1}{\frac{3}{5}}$, the reciprocal of a number is generally defined as the quotient after dividing 1 by the number; *e.g.* $\frac{1}{6}$ is the reciprocal of 6.

3. If £750 is $\frac{3}{5}$ of the value of a house, the house is worth $\frac{5}{3}$ of £750.

4. Again, if the net price is $\frac{230}{240}$ of the nominal price, the nominal price is $\frac{240}{230}$ of the net price.

5. To apply this. If I pay 7d. in the £ income tax; instead of 240 pence I receive 233 pence, therefore the actual income is $\frac{233}{240}$ of the nominal, and the nominal is $\frac{240}{233}$ of the actual income.

6. What is the number of which $\frac{3}{5}$ is 213? The number of course is $\frac{5}{3}$ of $213 = 357$.

7. What number multiplied by $\frac{4}{7}$ gives 516? Here again the number is found by finding $\frac{7}{4}$ of 516, or 903.

8. £100 is divided amongst A, B, C, D. A's share is $\frac{3}{7}$ of B's, and B's $\frac{4}{9}$ of C's, and D's $\frac{5}{9}$ of A's. What will each have?

We can take either's share as the unit with which to work.

Let us take B's share as our unit.

A's share is	$\frac{3}{7}$	of this unit.
B's "	1	"
C's "	$\frac{9}{4}$	"
D's "	$\frac{5}{9}$ of $\frac{3}{7}$, or $\frac{5}{21}$	"

$\therefore$ altogether there were $\dfrac{36+84+189+20}{7\cdot4\cdot3}$ units,

and £100 ÷ this fraction

$$= £\frac{100}{1} \times \frac{\cancel{7}\times 4\times 3}{\cancel{329}\,47} = £\frac{1200}{47} = £25\frac{25}{47}$$

= the unit or B's share—

$$\text{A's share is } \frac{3}{7} \text{ of } £\frac{1200}{47} = \frac{3600}{329} = £10\frac{310}{329}.$$

$$\text{C's share is } \frac{9}{\cancel{4}} \text{ of } £\frac{\overset{300}{\cancel{1200}}}{47} = \frac{2700}{47} = £57\frac{21}{47}.$$

$$\text{D's share is } \frac{5}{\underset{7}{\cancel{21}}} \text{ of } £\frac{\overset{400}{\cancel{1200}}}{47} = \frac{2000}{329} = £6\frac{26}{329}.$$

We might save fractions at the beginning of our work by assuming $\dfrac{1}{7\cdot4\cdot3}$ of B's share as our unit.

Then A had 36 such units.
B „ 84 „
C „ 189 „
D „ 20 „
or all four had 329 units;

$\therefore$ each unit was worth £$\dfrac{100}{329}$;

$$\therefore \text{A had } \frac{36\times100}{329} = £10\frac{310}{329}.$$

etc. etc. as before.

Had we expressed the other shares in terms of C's, we should have had—

A's share = $\frac{3}{7}$ of $\frac{4}{9} = \frac{4}{21}$.
B's share = $\frac{4}{9}$.
C's share = 1.
D's share = $\frac{5}{9}$ of $\frac{4}{21} = \frac{20}{189}$.

And if to avoid fractions we had assumed as our unit a fraction of C's share, it would have been $\frac{1}{3\cdot7\cdot3\cdot3}$, but A would still have had 36 units, B 84, etc.

9. Here is a question involving reciprocals.

If, at billiards, A can give B 20 points out of 100, and B can give C 20, and C can give D 20. What ought A to give D in a game of 1000?

At first sight some might think A could give D 60 out of a hundred, but it is not so.

A can make 100 points whilst B makes 80; $\therefore$ A's play is $\frac{100}{80}$, or $\frac{5}{4}$ of B's. Similarly B's is worth $\frac{5}{4}$ of C's, and C's is worth $\frac{5}{4}$ of D's;

$\therefore$ A's is worth $\frac{5\cdot5\cdot5}{4\cdot4\cdot4}=\frac{125}{64}$ of D's, or D's play is worth $\frac{64}{125}$ of A's play. Whilst A therefore is making 1000 points, D can only make $\frac{64}{125}$ of 1000, or 512 points; hence, to make the game fair, A ought to give D 488 points.

10. Here is a question which was set in a previous examination at Cambridge some few years ago.

If water increase in volume one-tenth when it is converted into ice, by what fraction does ice decrease when reconverted into water?

Volume of ice $=1\frac{1}{10}$, or $\frac{11}{10}$ of the volume of the water; hence by reciprocals, the volume of the water is $\frac{10}{11}$ of the volume of the ice, or ice decreases by one-eleventh when reconverted into water.

11. A right angle in England is divided into 90 equal parts called degrees, but in France it is divided into 100 equal parts called grades. Hence to convert degrees into grades we must multiply the number of degrees by $\frac{100}{90}$, or $\frac{10}{9}$, or $1\frac{1}{9}$, that is, there are $\frac{1}{9}$ as many again grades as there are degrees in any angle. By what fraction is the number of grades decreased if altered into degrees? Since the number of grades = ten-ninths of the number of degrees in any angle, the number of degrees is $\frac{9}{10}$, or one-tenth less than the number of grades.

12. If the betting against a horse be (say) 8 to 3, out of 11 chances it has 3, or $\frac{3}{11}$ of certainty. If there are 2 horses

only, of course the other has 8 chances, and the betting is 8 to 3 *on* it.

Supposing there are 3 horses, one of which must win, and the odds are 6 to 1 against one and 3 to 1 against the second. What is fair betting about the third? The first horse has 1 chance out of 7, or $\frac{1}{7}$ of certainty, and the second $\frac{1}{4}$ of certainty. Now we must reduce these to the same denomination and add them, and the third horse's chance will be expressed by the difference between this sum and 1, viz. $\frac{28-(4+7)}{28}$ or $\frac{17}{28}$; that is, 17 to 11 on it, or 11 to 17 against it.

Examination and Examples.

1. $\frac{3}{64}$ of a ship is worth £384, what is the value of the whole ship and $\frac{7}{12}$ of it?

2. If $\frac{103}{100}$ of the buying price be the selling price, what did I pay for that which I sold at £1000?

3. A creditor receives $\frac{37}{80}$ of his debt, and afterwards $\frac{3}{16}$ of the remainder. What did he lose on a debt of £592?

4. If the betting *against* one horse out of two in a race be 5 to 3, what is the betting *on* the other?

5. At billiards A can give B 40 points out of a hundred, and B can give C 40 points out of a hundred. How many points ought C to receive from A in one thousand?

6. A has half as much again as B, and a third as much again as C. What is C's of B's?

7. If only 2 horses are in for a race, and the betting on one is 14 to 9, what is the fair betting with regard to the other?

8. If out of 3 horses the betting is 2 to 1 against one, and 3 to 7 on another, what is the betting with regard to the third?

9. If I have a third as much again as my brother, how much less has my brother than I?

10. Of what number is 18 ten less than its half?

11. If A has $\frac{1}{3}$ as much again as B, and B $\frac{1}{4}$ as much again as C. What has C of A's?

12. A has $\frac{1}{2}$ as much again as B and C, and B has $\frac{1}{5}$ as much again as C. What has C of A's?

13. In an election A receives promises from half the constituency, B from $\frac{19}{40}$; 400 voters, however, had promised both candidates, of whom 300 vote eventually for B and 100 for A. The electors who had not promised either, do not vote. B wins by 80 votes. What were the numbers for each?

14. Pipes A and B can fill a cistern in 3 minutes and 5 minutes respectively, and C can empty it in $7\frac{1}{2}$ minutes. In what time will the cistern be filled when A, B, and C are all open?

15. If 5 men and 9 boys could do a piece of work in 17 days, in how many days would 9 men and 12 boys do it; the work of 2 men being equal to that of 3 boys.

16. If 12 men working 8 hours a day do $\frac{4}{5}$ of a piece of work in 20 days, how many days will 15 men working 10 hours a day take to do $\frac{7}{8}$ of it?

17. A and B are engaged on a work which must be finished in 6 days. A works 8 hours a day at first, and B 9 hours; after 3 days only $\frac{1067}{2261}$ of the work is done, each therefore works 1 hour a day more, and it is just finished in time. Compare the rates of working of A and B.

18. If two boats, A and B, row in a race at their usual speed, A will win by 80 yards; but the day proving unfavourable, A can only row at $\frac{8}{9}$ of its usual speed, while B rows at $\frac{9}{10}$ of its usual speed. A wins by 26 yards. Find the length of the course.

19. Twelve men are engaged to do a piece of work which they would complete by working 8 hours a day for 14 days; and when half the work is done 8 of the men leave. In what number of days will the rest be able to complete it, if they work $10\frac{1}{2}$ hours each day?

20. Two elephants and 4 horses can do a piece of work in $7\frac{1}{7}$ hours, and a single elephant can do it all in 20 hours. How long will a single horse take in doing it?

21. If £90, the selling price, be $\frac{6}{5}$ of the buying price, what profit do I make on the transaction?

22. The circumferences of two wheels of a locomotive are 5 and 6 yards respectively. If the smaller make 704 revolu-

tions in a given time, when the engine travels at the rate of 40 miles an hour, how many revolutions will the larger make in the same time, when the engine travels at the rate of 60 miles an hour.

23. Five town workmen can do the same amount of work as 9 country workmen. What would be the difference in expense to a contractor between employing 15 town workmen, or 10 town and 12 country workmen, to do a piece of work which the 15 town workmen could finish in 10 days, the wages of a town and country workman being respectively 7s. 6d. and 4s. 6d. a day?

24. On a piece of work 3 men and 5 boys are employed, who do half of it in 6 days. After this one more man and one more boy are put on, and $\frac{1}{3}$ more is done in 3 days. How many more men must be put on that the whole may be completed in one day more? See No. 24, page 159.

25. Three men, A, B, C, play a game starting with equal sums; each stakes for each game $\frac{1}{3}$ of all he then has. If A wins the first game and B the second, what fraction of what they originally had has each left?

26. A cistern is supplied with two feeding pipes, capable of filling it in half an hour and three-quarters of an hour respectively; and one discharge pipe capable of emptying it in a quarter of an hour. The cistern is full, and the three pipes are set in action together. What portion of the cistern will remain filled in three-quarters of an hour?

27. If the price of candles $8\frac{1}{2}$ in. long be 9d. per half-dozen, and that of candles of the same thickness and quality $10\frac{1}{4}$ in. long be 11d. per half-dozen, which kind do you advise a person to buy?

28. A quantity of coffee has chickory added to it, so that the chickory is $\frac{1}{9}$ of the whole mixture. If the mixture be worth 2s. per lb. and the chickory 4d. per lb., what is the exact value per lb. of the pure coffee?

29. If two horses are worth as much as 5 oxen, and 3 oxen as much as 16 sheep, what will be the value of a horse if the price of a sheep is £3, 15s.?

30. What decimal fraction of £1 must be added to $\cdot\dot{0}\dot{9}$ of £6, 13s. $4\frac{1}{2}$d. that the result may be 15s.?

31. I purchase books at $\frac{5}{6}$ of the selling price, but I return 1d. in the shilling on the marked price. What profit do I make on books which cost me £300?

32. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days. In what time could each alone complete the work?

33. A grazier spent £33, 7s. 6d. in buying sheep of different sorts. For the first sort, which formed one-third of the whole, he paid 9s. 6d. each. For the second sort, which formed one-fourth of the whole, he paid 11s. each. For the rest he paid 12s. 6d. What number of sheep did he buy?

34. Two persons, A and B, have the same income. A lays by one-fifth of his; but B, by spending £60 per annum more than A, at the end of three years finds himself £100 in debt. What is the income of each?

35. Give yourself the answer and the other data except the amount of B's debt, and find it.

36. Give yourself the answer and the other data except the £60 which B spends more than A, and find it.

37. Two casks, A and B, contain mixtures of wine and water; in A the quantity of wine is $\frac{4}{3}$ of the quantity of water; in B the water is $\frac{3}{2}$ of the wine. If A contain 84 gallons, what must B contain so that when the two are put together the new mixture may be half wine and half water?

38. A pudding consists of 2 parts of flour, 3 parts of raisins, and 4 parts of suet; flour cost 3d. a lb., raisins 6d., and suet 8d. Find the cost of the several ingredients when the whole cost 2s. 4d.

39. A watch gains as much as a clock loses, and 1799 hours by the clock are equivalent to 1801 hours by the watch. Find how much the watch gains and the clock loses per hour.

40. I lost $\frac{1}{3}$ of my property in an unfortunate speculation, and then had £1562, 3s. 4d. left. How much did I lose, and how much had I at first?

41. If the cost price be $\frac{91}{100}$ of the selling price, find the selling price of that which cost £273.

42. If the selling price be $\frac{111}{100}$ of the cost price, find the cost price of goods sold for £259.

43. If $\frac{3}{5}$ of a cask of brandy and water be brandy, how much water is there in a cask of 360 gallons?

44. If A's pace is $\frac{3}{5}$ of B's, how long will it take B to walk the distance A can walk in 4 hours?

45. Four thalers, 12 dollars, 10 florins, and 3 guineas amount to £7, 2s. If the value of a dollar be $\frac{27}{50}$ of a thaler, find their value.

46. A grocer buys a chest of tea containing 180 lbs. at 3s. $7\frac{1}{4}$d. per lb. If 10 lbs. be spoiled, what does he gain by selling the remainder at 4s. 4d. per lb.?

47. In an event between 3 candidates, if the chances are 10 to 1 against the first and 2 to 1 against the second, show that the chances for the third are 19 to 14.

48. A sold an article to B at a loss of 2d. per shilling on the outlay, but B selling it again to C at 3s. 9d. gained $1\frac{1}{2}$d. on his shilling of outlay. What did the article cost A?

49. A waterman found by experience that with the advantage of a common tide he could row down a river from A to B, which is 18 miles, in $1\frac{1}{2}$ hours; and that to return from B to A during the same tide, though he rowed back along the shore, where the stream was only three-fifths as strong as in the middle, took him $2\frac{1}{4}$ hours. Required the hourly velocity of the middle current.

50. If £5, 6s. 8d. (the selling price) is $\frac{112}{100}$ of the buying price, at what price did I buy the article?

CHAPTER IX.

Recurring Decimals and other Radix Fractions—Their Reduction—Duodecimals treated as Radix Fractions.

1. As stated in Chapter vii. par. 12, if in the denominator of a fraction (in its lowest terms) there is any factor other than 2 or 5, the fraction cannot be reduced to a terminating decimal, since of no other numbers than 2 and 5 is 10, or 100, or 1000, etc., a multiple.

Similarly in other radix fractions, those fractions (in their lowest terms) whose denominators contain any factors except those which are themselves factors of the radix itself, can be reduced to terminating decimals.

2. In reducing a vulgar fraction to its corresponding decimal fraction, whenever the remainder becomes the same as one we have had before (provided, of course, that we have to bring down a cipher or the same figure from the dividend), the figures in the quotient will repeat, which is expressed by means of a dot or dots placed over the first and last of such figures. Thus—

$$\frac{3}{7} = 0 \cdot \dot{4}2857\dot{1}, \text{ since } \begin{array}{l} 7)\underline{3 \cdot 000000} \\ \quad 0 \cdot 428571 \\ \quad\;\; 3264513. \end{array}$$

Here we have placed the remainders under the corresponding figures of the quotient.

3. Reduce (by Short Division) $\frac{4}{21}$ to a recurring decimal.

We can either divide by the 3 or 7 first. Let us perform the operation both ways side by side—

$$\begin{array}{ll} 3)\underline{4\cdot000000} & \qquad 7)\underline{4\cdot000000} \\ 7)\underline{1\cdot3333333}\text{, etc.} & \qquad 3)\underline{\cdot571428571}\text{, etc.} \\ 0\cdot190496 & \qquad 0\cdot190476 \\ \;\;1603541 & \qquad \;\;201210, \end{array}$$

whence we see the decimal is $\cdot\dot{1}9047\dot{6}$.

4. If we know that $\frac{3}{7}=\cdot\dot{4}2857\dot{1}$, we can write down the value of all fractions whose denominator is 7; thus—$\frac{6}{7}=\cdot\dot{8}5714\dot{2}$. The first figure is evidently 8, hence we only have to begin with 8 (from the equivalent decimal to $\frac{3}{7}$) and write six figures. Similarly $\frac{2}{7}=\cdot\dot{2}8571\dot{4}$.

5. Since the figures in the quotient will repeat themselves when the remainder is the same as any remainder we have had before, and there are only six possible remainders after dividing by 7, viz. 1, 2, 3, 4, 5, or 6, there can only be six figures (or one less than the number 7) in the period. So in the decimal corresponding to the vulgar fraction $\frac{5}{17}$, there can only be 16 figures in the period.

6. Though there *can* only be 12 figures in the period of the recurring decimal which is equivalent to $\frac{5}{13}$, there need not be as many, nor in fact are. In $\frac{2}{3}$ there *might* be two figures, but there is only one in the period of the corresponding decimal.

7. Reduce $\frac{2}{9}$ to a radix fraction (undenary).

$$\begin{array}{l} 9)\underline{2\cdot000000} \\ 0\cdot249861 \\ \;\;2487512 \end{array}$$

hence the radix fraction is $\cdot\dot{4}8751\dot{2}$. Knowing this, we might immediately write down $\frac{5}{9}$ or $\frac{7}{9}$, thus—

$$\frac{5}{9}=\cdot\dot{6}1249\dot{8} \text{ and } \frac{7}{9}=\cdot\dot{8}6124\dot{9}.$$

8. To convert a recurring decimal to a vulgar fraction, we have really to sum an infinite series.

Thus $\cdot\dot{2}\dot{3}$ means $\frac{23}{100}+\frac{23}{10000}+\frac{23}{1000000}+$etc., *ad infinitum*, and Algebra teaches us that this is equal to $\frac{23}{99}$.

9. Since $\frac{1}{9}=\cdot\dot{1}$, and $\frac{2}{9}=\cdot\dot{2}$, etc., to convert a recurring decimal with one figure in the period, as $\cdot\dot{7}$, we may do so by making the period our numerator and 9 our denominator.

Again, $\frac{1}{99}=\cdot\dot{0}\dot{1}$, $\frac{17}{99}=\cdot\dot{1}\dot{7}$, or any fraction with 99 for its

denominator will reduce to a recurring decimal with two figures in the period.

Again, $\frac{1}{999} = \cdot\dot{0}0\dot{1}$, or $\frac{375}{999} = \cdot\dot{3}7\dot{5}$, so that we venture to form a rule, that if all the decimal places recur, we may take them for the numerator, and that for the denominator we must have as many nines as there are figures.

$$\text{Hence } \cdot\dot{4}71\dot{6} = \tfrac{4716}{9999},\ \cdot\dot{0}006\dot{7} = \tfrac{67}{99999}.$$

10. Note that this reason for the rule given in paragraph 9 is not rigidly proved there; still, if it be true, we may rigidly prove the rule for reducing a mixed recurring decimal to its equivalent vulgar fraction.

Let us have to reduce the decimal fraction $3\cdot2\dot{7}1\dot{5}$ to its equivalent vulgar fraction.

$$3\cdot2\dot{7}1\dot{5} = \frac{32\cdot\dot{7}1\dot{5}}{10} = \frac{32\frac{715}{999}}{10}$$

$$= \frac{32\frac{715}{1000-1}}{10} = \frac{32000-32+715}{\frac{\frac{999}{10}}{1}}$$

$$= \frac{32715-32}{9990} = \frac{32683}{9990},$$

whence the rule given at p. 106.

11. As we wish students to reduce radix fractions to their equivalent fractions, we venture to give here the ***algebraical*** method of reducing them.

Let $34\cdot2\dot{1}\dot{7}$ be our radix fraction in the scale of 8.

Let $V = 34\cdot21717171717$, etc.

Multiply by 1000 or 512 (denary).

$1000V = 34217\cdot171717$, etc.

Multiply by 10 or 8.

$10V = 342\cdot171717$, etc.

Subtracting,

$$770V = 33655, \text{ or } V = \tfrac{33655}{770}.$$

In this we first multiplied by 1000, so as to move the unit point to the end of the first period. We next multiplied by 10, so as to move the unit point to the beginning of the first period. In this way we obtained two radix fractions of which the fractional parts were the same, viz. ·1717, etc.

Had the radix been undenary, we should have obtained for our denominator *tto*, since 1000 − 10 in the undenary scale leaves *tto*.

12. Since $\cdot\dot{2}\dot{3} = \cdot\dot{2}3232\dot{3}$, as they both mean the same thing, and $\cdot\dot{4}1\dot{7} = \cdot\dot{4}1741\dot{7}$ for the same reason,

$$\therefore \frac{\cdot\dot{2}\dot{3}}{\cdot\dot{4}1\dot{7}} = \frac{\cdot\dot{2}3232\dot{3}}{\cdot\dot{4}1741\dot{7}} = \frac{\dfrac{232323}{999999}}{\dfrac{417417}{999999}} = \frac{232323}{417417}$$

$$= \frac{33189}{59631} = \frac{11063}{19877}.$$

13. Though recurring decimals and radix fractions had better be added in the form of decimals or radix fractions, they must be reduced to vulgar fractions before being multiplied or divided.

Thus divide $4\cdot1\dot{7}$ by $2\cdot\dot{3}$ both in the octonary.

$$\frac{417-41}{\underset{10}{\cancel{7}\cancel{0}}} \times \frac{\cancel{7}}{23-2} = \frac{356}{210} = 1\cdot6.$$

14. Even in arithmetics where scales of notation are not introduced, duodecimals are generally explained in connection with feet and inches. When we come to the mensuration of planes and solids, we shall seem to be multiplying feet and inches by feet and inches. Since 12 in. make 1 ft., if we multiply 3 ft. 4 in. by 4 ft. 3 in., and the result by 5·7 in., the operation will be the same as multiplying 3·4, 4·3, and 5·7 altogether in the duodenary scale, thus—

3·4 ft.
4·3 ft.

---

*t*0
114

12·20 sq. ft.
5·7 ft.

832
5*tt*

67·12 cubic ft.,

which represents 79 cubic ft. + $\frac{1}{12}$ of 1728, or 144 cubic in., and $\frac{2}{144}$ or 24 cubic in.,

that is, 79 cubic ft. 168 cubic in.

This working compared with the ordinary method will be found to be very much shorter.

At this point we only wish the student to look upon questions of this kind as problems in the duodenary scale. When we come to the measuring of surfaces and solids, we will discuss the rules and the terms used in this branch of the science as usually taught in text-books.

15. All questions will be given in this form—Reduce to feet and radix fractions of a foot in the duodenary scale, 4 cubic yds. 17 ft. 769 in., and 1 yd. 1 ft. 8 in., and divide the cubic feet by the linear feet of which the quotient will be square feet.

4 cubic yds. 17 ft. 769 in. = 125 cubic ft. 769 in.

To reduce this to duodenary.

12)125	12)769
12)10 . . . 5	12)64 . . . 1
0 . . . *t*	5 . . . 4;

∴ 125 denary = *t*5 duodenary;

∴ 769 cubic in. = 541 duodenary.

But five 144 in. = $\frac{5}{12}$ cubic foot,

and four 12 in. = $\frac{4}{144}$ cubic foot,

and 1 in. = $\frac{1}{1728}$ cubic foot.

Or 541 (duodenary) cubic in. = ·541 (duodenary) cubic ft., and 1 yd. 1 ft. 8 in. = 4·8 (duodenary) feet.

$$\begin{array}{l}
4{\cdot}8'\,\big)\,t5{\cdot}5'41\,\big(\,22{\cdot}t6t6 \\
\qquad\;\; 94 \\
\qquad\;\; \overline{115} \\
\qquad\;\;\; 94 \\
\qquad\;\;\; \overline{414} \\
\qquad\;\;\; 3t8 \\
\qquad\;\;\;\; \overline{281} \\
\qquad\;\;\;\; 240 \\
\qquad\;\;\;\;\; \overline{410} \\
\qquad\;\;\;\;\; 3t8 \\
\qquad\;\;\;\;\;\; \overline{240} \\
\qquad\;\;\;\;\;\; 240 \\
\qquad\;\;\;\;\;\; \overline{\;\cdots\;}
\end{array}$$

Result: 26 sq. ft. $126\frac{126}{144}$ in.

The 8 in the divisor and 5 in the dividend are marked to show where the unit point is to be placed in the quotient.

The 22 (duodenary) feet is, of course, equal to $2 \times 12 + 2$, or 26 (denary) feet.

The *t*6, which immediately succeed the unit point, is = 126 one hundred and forty-fourths of a foot, or 126 inches, and the other *t*6 are 126 over 144×144 of a foot, or so many hundred and forty-fourths of an inch.

The result could, of course, be reduced to 2 sq. yds. 8 ft. $126\frac{7}{8}$ in., but in ordinary duodecimals would be given as

$$22 \text{ ft. } 10^{\text{i.}}\ 6^{\text{ii.}}\ 10^{\text{iii.}}\ 6^{\text{iv.}},$$

the notation of which will be explained later on.

16. Reduce to radix fraction (octonary) of a pound,—£4, 7s. $7\frac{1}{2}$d.

$$
\begin{array}{l}
2\,\underline{)1\cdot} \text{ halfpenny.} \\
12\,\underline{)7\cdot4} \text{ pence.} \\
20\,\underline{)7\cdot5} \text{ shillings.} \\
\pounds 4\cdot364.
\end{array}
$$

Here the operation is exactly the same as in the denary scale, multiplying the remainders by 8 instead of 10.

Examination and Examples.

1. State clearly what you understand by $\cdot\dot{3}\dot{4}$.
2. Reduce $1\cdot\dot{3}\dot{4}$ to a vulgar fraction.
3. Divide $1\cdot\dot{1}\dot{5}$ by $5\cdot7$.
4. Multiply $1\cdot11$ by $\cdot2\dot{3}$.
5. Find the difference between $1\cdot0\dot{6}$ and $1\cdot06$, and reduce the answer to a vulgar fraction.
6. What number, multiplied by $\cdot42\dot{6}$, will give a product $2\cdot1\dot{3}$?
7. State clearly what you understand by $\cdot\dot{3}\dot{4}$ (septenary).
8. Reduce $\cdot\dot{3}\dot{4}$ septenary to a vulgar fraction expressed in same scale.
9. Reduce $4\cdot7\dot{1}\dot{2}$ (octonary) to vulgar fraction expressed in same scale.
10. Reduce $5\cdot\dot{7}$ (nonary) and $\cdot2\dot{3}$ (nonary) to vulgar fractions, and divide the former by the latter. Reduce the quotient to a radix fraction.
11. Evaluate $3\cdot1\dot{7}$ of £2, 6s. 3d.
12. Divide £5, 19s. 2d. by $\cdot4\dot{9}$.
13. Reduce $3\cdot17\dot{4}$ (octonary) to a vulgar fraction, reduce its terms to the denary, and change it into a decimal fraction.
14. Why is the term 'decimal point' an inadequate name?
15. If I move the unit point in a radix fraction (septenary) three places to the right, what operation have I performed?
16. If I move the unit point in a radix fraction (octonary) three places to the left, what operation have I performed?
17. Express as a £ in a radix fraction (nonary) £2, 6s. 4d.
18. Evaluate £$3\cdot1412$ (quinary).

19. Express as a £ decimally $3\frac{1}{2}$d.

20. If money increases 1·05 of itself each year, what is £200 worth at the end of 4 years?

21. If money increases by 4 hundredths each year, what is the value of £300 at the end of 3 years?

22. If money decreases by 2 hundredths every 6 months, what is the value of £180 in 2 years?

23. Write down some vulgar fractions that will reduce to terminating radix fractions (duodenary).

24. What fraction is greater by $\cdot\dot{5}\dot{7}$ than the product of $\cdot\dot{2}$ and 2·3?

25. How many times can $\cdot 3\dot{7}$ be subtracted from $11\cdot\dot{2}$, and what is the denomination of the remainder?

26. Evaluate 3·1675 (octonary) tons.

27. Find some sum of money which will reduce to a terminating radix fraction of £ (octonary).

28. If my selling price is $1\cdot 0\dot{3}$ of my buying price, what profit do I make on £1000?

29. If my buying price is $\cdot 8\dot{7}$ of my selling price, what do I gain on goods for which I receive £1000?

30. If my buying price is $\cdot 8\dot{5}$ (nonary) of my selling price, what do I gain on that for which I receive £729 (denary)?

31. If my selling price is 1·16 of my buying price, what do I gain on what costs me £112, 10s.?

32. If I pay in wages $\cdot\dot{3}$ of my gross receipts, and another $\cdot\dot{3}$ for my raw material, and $\cdot\dot{1}$ for my rent and taxes, and my profits are £1000, what are my gross receipts?

33. Give yourself the result of 32 and all the data except the fraction of your gross receipts that you pay for raw material, and find it.

34. If I pay $\cdot 0\dot{1}$ of my property to insure it against fire, and this comes to £4, 2s. 6d., what is the value of my property?

35. If I pay a broker ·00125 of the money I invest for his work, and his fee come to £125, 17s. 6d., what money do I invest?

36. If for investing £12500 I pay my broker £15, 12s. 6d., what decimal of the whole do I pay him?

37. Of an estate $\cdot\dot{2}\dot{7}$ are left to the widow, $\cdot\dot{1}4285\dot{7}$ to each of three daughters, and the rest to two sons so that the elder

has twice what the younger has. Find the younger son's share if the estate be worth £1663, 4s.

38. Divide $376\cdot\dot{5}$ amongst A, B, and C, so that B has double what C has, and A double what B has.

39. Multiply $3\cdot\dot{6}$ by $\cdot\dot{3}6$, and divide result by $\cdot 03\dot{6}$.

40. How often is $3\cdot 6$ shillings contained in £$12\cdot 14\dot{6}$?

41. Reduce $35\cdot 1\dot{7}$ octonary to denary.

42. Divide $4\cdot\dot{7}$ (nonary) by $\cdot 00\dot{3}$ (nonary), giving your answer in the nonary.

43. Reduce $2\cdot\dot{5} \div \cdot 4\dot{7}$ (both denary) to the octonary.

44. Change £$5\cdot 7314$ (nonary) to £, s. d.

45. Express £2, 6s. 8d. to the radix fraction of £3 (nonary).

46. Reduce $37\cdot 1\dot{4}\dot{5}$ (duodenary) to a vulgar fraction expressed in the duodenary.

47. Reduce $23\cdot 3\dot{5}$ senary to a vulgar fraction expressed in the denary.

48. Reduce 3grs. to the radix fraction (quinary) of a lb. apoth.

49. Reduce 55 sq. in. (duodenary) to acres.

50. Reduce to feet (duodenary) and multiply together (reducing your answer to the denary)—1 po. 1 yd. 1 ft. 1 in., 3 yds. 2 ft. 7 in., 5 yds. 2 ft. 11 in., and divide result by 2 sq. yds. 8 ft. 87 in.

APPENDIX TO CHAPTER IX.

To reduce a recurring decimal (or radix fraction) to a vulgar fraction, for the numerator subtract the non-recurring figures, both integers and decimals (or radix fractions), from the entire number, both integers and decimals (or radix fractions), and for the denominator put as many nines (or one less than the radix) as there are recurring figures, followed by as many ciphers as there are non-recurring *decimals* (or radix fractions).

To add or subtract recurring decimals (or radix fractions), carry them out to three or four more places than that to whose accuracy you are asked for, and add and subtract as in integers.

To multiply or divide recurring decimals (or radix fractions), reduce them to vulgar fractions, and multiply or divide, and reduce the resulting fraction back to a decimal (or radix fraction).

CHAPTER X.

Problems and Remarks on Foregoing Chapters.

1. A problem is sometimes seemingly made more difficult by the introduction of elements on which the solution does not depend.

Here is a difficult problem from a paper set at St. John's College, Cambridge.

Ash saplings after 5 years' growth are worth 1s. 3d., and increase in value 1s. 3d. each year afterwards. For their growth they require each twice as many square yards as the number of years they are intended to grow before cutting. A plantation is arranged so that each year the same number may be ready for cutting. Find the greatest annual income which can be obtained per acre, allowing $\frac{1}{5}$ of the gross receipts for expenses.

In this problem, as we are asked for a result per acre, and the number of trees in an acre is limited by the conditions of the problem, the fact that there are to be the same number of trees ready for cutting each year does not affect the question.

Though the price of the saplings increases each year, first we have to wait longer for our harvest, and secondly, we have to give the tree more room, and to cut fewer from each acre.

At *the end* of 5 years we should receive

$$\frac{4}{5} \text{ of } \frac{4840}{2 \times 5} \times 1\text{s. } 3\text{d.} \times 1 \text{ per acre,}$$

which gives us for each year $\frac{1}{5}$ of $\frac{4}{5}$ of $\frac{4840}{2 \times 5} \times 1\text{s. } 3\text{d.} \times 1$ per ac.

At *the end* of 6 years we should receive

$$\frac{4}{5} \text{ of } \frac{4840}{2 \times 6} \times 1\text{s. } 3\text{d.} \times 2 \text{ per acre,}$$

which gives us for each year $\frac{1}{6}$ of $\frac{4}{5}$ of $\frac{4840}{2\times 6}\times$ 1s. 3d. $\times$ 2 per ac.

Of the elements in these two expressions we find the $\frac{4}{5}$ of $\frac{4840}{2}$ and 1s. 3d. unaltered; but the denominator 5 in $\frac{1}{5}$, the 5 under the 4840, and the factor 1 at the end, all increased by 1.

We have then, according to the number of years the saplings are to grow, these multipliers of the unchanging quantity $\frac{4}{5}$ of $\frac{4840}{2}\times$ 1s. 3d.

$$\begin{array}{llll}
\text{For 5 years} & \frac{1}{5\times 5}\times 1, \text{ or } \frac{1}{25}. & & \\
\text{,, 6 ,,} & \frac{1}{6\times 6}\times 2=\frac{1}{18}. & & \\
\text{,, 7 ,,} & \frac{1}{7\times 7}\times 3=\frac{3}{49}, \text{ or } \frac{1}{16\frac{1}{3}}. & & \\
\text{,, 8 ,,} & \frac{1}{8\times 8}\times 4, \text{ or } \frac{1}{16}. & & \\
\text{,, 9 ,,} & \frac{1}{9\times 9}\times 5, \text{ or } \frac{5}{81}, \text{ or } \frac{1}{16\frac{1}{5}}. & & \\
\text{,, 10 ,,} & \frac{1}{10\times 10}\times 6, \text{ or } \frac{3}{50}, \text{ or } \frac{1}{16\frac{2}{3}}. & & \\
\text{etc.} & \text{etc.} \quad \text{etc.} & &
\end{array}$$

And examining these fractions, whose numerators are all 1, we see directly that they increase up to that for 8 years, and afterwards decrease. We leave the student to find the result, viz. £7, 11s. 3d.

2. Find the largest number that will divide 475 and 599 and leave a remainder 3. All that we have to do is to subtract the 3 from these two numbers and find the G. C. M. of the remainder.

3. Here is another problem from a St. John's paper which is worthy of notice.

A contractor employs a fixed number of men to complete a work. He may employ either of two kinds of workmen; the first at 26s. 6d. per week each, the second at 18s. 6d. per week each,—the work of 5 of the latter men being only equal to 4 of the former. If he finish it as quickly as possible, he spends £270 more than he would have done if he had

finished it as cheaply as possible, but takes 4 weeks less time. What would it have cost if he had employed equal numbers of the two kinds of workmen.

Had the inferior workmen been able to do as much work per week as the better, they would have earned each week $\frac{5}{4}$ of 18s. 6d., or 23s. 1½d.; and the difference between 26s. 6d. and 23s. 1½d. is 3s. 4½d., which is contained in £270 1600 times; ∴ the work was one of 1600 weeks for one of the better men, and a work of 2000 weeks for one of the inferior men.

If an equal number of men are employed to finish it together.

The better men must do $\frac{5}{9}$ of the work, or $\frac{5 \times 1600}{9}$, which cost $\frac{5 \times 1600}{9} \times$ 26s. 6d.

The inferior men do the remaining $\frac{4}{9}$ of the work, or $\frac{4 \times 2000}{9}$, which costs $\frac{4 \times 2000}{9} \times$ 18s. 6d.;

$$\therefore \text{ the cost is } £\frac{5 \times 1600}{9} \times \frac{26\frac{1}{2}}{20} + \frac{4 \times 2000}{9} \times \frac{18\frac{1}{2}}{20}$$

$$= £\frac{8000}{9} \times \frac{45}{20} = £2000.$$

In this problem we did not require to know the number of men employed, and therefore it was not necessary to take into consideration the fact that the slower workmen would take 4 weeks longer.

This, then, is really an element introduced without any necessity, being one on which the solution of the question asked does not depend. Had the number of men been asked for, then we should have required to know some other fact beyond the difference in cost.

If time alone had been asked for, then the answer could have been found without reference to the price.

To find how many men were employed here, we must introduce this second condition, viz. that the inferior men would take 4 weeks longer.

Whilst, then, the better men are doing the work, the inferior could only do $\frac{4}{5}$ of it, and since it takes them 4 weeks to do the other fifth, it takes them altogether 20 weeks to do it, and there are 2000 of their weeks' work to do; hence 100 men must have been employed.

This problem might be varied in many ways.

4. To prove that all prime numbers greater than 3 are within 1 of multiples of 6.

Since all numbers are within 1 of multiples of 3, and a prime number greater than 2 must be odd, therefore the multiple of 3 nearest a prime number must be an even number; hence a multiple of 6.

5. The converse of this is not true. In fact, no expression has or can be found to express nothing but prime numbers. *Generally* every multiple of 6 has a prime number within 1 of it, but *not always; e.g.* 145, 147 are neither of them prime, though within 1 of 144, nor 119 or 121.

6. Erastothene's sieve for finding prime numbers is this. Write down all the odd numbers from 3, begin with 3 and cut out every third figure from it, which, of course, will be these—9, 15, 21, etc.; then begin with 5 and cut out every fifth figure (counting those already cut out), which of course will cut out the 15, 25, 35, etc.; then begin with 7 and cut out every seventh, and so on, and the remaining figures will be prime.

7. To compare fractions. They can either be reduced to a common denominator, and the numerator compared or reduced to a common numerator and the denominators compared. The simplest number to reduce the numerators to, is 1, which is done as shown in Chapter v., par. 23.

Compare the fractions $\frac{3}{7}$, $\frac{5}{11}$, $\frac{11}{23}$, $\frac{13}{27}$, $\frac{15}{31}$.

The L. C. M. of these fractions is a very large number, whereas they are immediately reduced to fractions whose numerators are 1, thus—

$$\frac{1}{2\frac{1}{3}},\ \frac{1}{2\frac{1}{5}},\ \frac{1}{2\frac{1}{11}},\ \frac{1}{2\frac{1}{13}},\ \frac{1}{2\frac{1}{15}},$$

whence we see that they are in their order of magnitude (the least being first), since $2\frac{1}{3}$ is greater than $2\frac{1}{5}$, and $2\frac{1}{5}$ than $2\frac{1}{11}$, etc.

8. Here is a question taken from a set of algebraical equa-

tions which can be solved by pure arithmetic. I divide a certain sum of money amongst 5 persons so that one has 7 less than half of it, the second 2 more than $\frac{2}{3}$ of the remainder, the third 8 less than $\frac{10}{17}$ of this remainder, the fourth 1 more than $\frac{1}{3}$ of this remainder, and the fifth has the remaining 9.

Beginning with the last division, the question to be solved is really this: What number after I have taken away 1 more than its third, will leave 9?

If I take away 1 more than $\frac{1}{3}$, I have 1 less than $\frac{2}{3}$; and if this be 9, the number must be $\frac{3}{2}$ of $(9+1)$, or 15. Secondly, I have to find the number from which, if I take away 8 less than $\frac{10}{17}$, I leave 15. Here I have 8 more than $\frac{7}{17}$; $\therefore$ the number is $\frac{17}{7}$ of $(15-8)$, or 17. Thirdly, I have to find the number from which, if I take away 2 more than $\frac{2}{3}$, I leave 17; which will be found to be $3\times(17+2)$ or 57. And the original number of pounds will be $2\ (57-7)$ or 100.

The example No. 25 is a very easy one of this sort.

It would be bad arithmetic (if arithmetic at all) to assume any unit here, as it would be impossible to do the working of the question entirely in this unit, as another unit, viz. pounds, enters.

9. How often can $\frac{1}{7}$ be subtracted from $3\frac{1}{6}$? and explain the nature of the remainder. In questions of this kind, what we are really asked is how many *whole* times can $\frac{1}{7}$ be subtracted from $3\frac{1}{6}$, and with what remainder. If we divide $3\frac{1}{6}$ by $\frac{1}{7}$, according to the ordinary method we get the entire quotient,—fraction and all,—and are told nothing about the remainder;

$$\text{thus, } 3\tfrac{1}{6}\div\tfrac{1}{7}=\tfrac{19}{6}\times\tfrac{7}{1}=\tfrac{133}{6}=22\tfrac{1}{6}.$$

This, of course, gives us the number of whole times, viz. 22, but does not give us the remainder, the $\frac{1}{6}$ being the fraction of a time the $\frac{1}{7}$ is contained in the remainder. The question then becomes, in what number is $\frac{1}{7}$ contained $\frac{1}{6}$ of a time? and the answer is $\frac{1}{42}$, therefore the remainder is $\frac{1}{42}$.

If we reduce these abstract numbers to concrete quantities by attaching them to a concrete unit, as £1, we see at once what our remainder must be—

$$£3\tfrac{1}{6}\div £\tfrac{1}{7} \text{ is } £3,\ 3\text{s. } 4\text{d.}\div 2\text{s. } 10\tfrac{2}{7}\text{d.};$$

reducing these in the ordinary method to sevenths of a penny—

£ s. d.	s. d.
3 3 4	2 $10\frac{2}{7}$
20	12
63 shillings	34 pence
12	7
760 pence	240 sevenths of a penny.
7	
5320 sevenths of a penny.	

$$
\begin{array}{r}
240 \text{ sevenths of a penny})5320 \text{ sevenths of a penny}(22 \text{ times.} \\
480 \\
\hline
520 \\
480 \\
\hline
40 \text{ sevenths of a penny.}
\end{array}
$$

Ans. 44 times and 40 sevenths of a penny over.

$$\frac{40}{7}\text{d.} = £\frac{40}{7.12.20} = £\frac{1}{42}.$$

Again :—1 seventh)19 sixths(is the same as
6 forty-seconds)133 forty-seconds(22 times
which gives quotient 22 with 1 forty-second as remainder.

In answering these questions in decimals we must stop dividing when we have found our units in the quotient.

How many times can 3·7 be subtracted from 114·81? When we divide the 8 we obtain units. So we proceed no farther in the division, and the remainder must be stated as such.

$$
\begin{array}{r}
3{\cdot}7')114{\cdot}8'1(31 \text{ units.} \\
111 \\
\hline
38 \\
37 \\
\hline
11
\end{array}
$$

Ans. 31 times and 11 hundredths over.

Examinations and Examples.

1. If 1 lb. of metal, of which $\frac{1}{4}$ is zinc and the rest copper, be mixed with 2 lbs. of which $\frac{1}{5}$ is zinc and the rest copper, how much copper is there in the mixture?

2. A person buys 8 lbs. of tea and 3 lbs of sugar for £1, 2s., and at another time he buys 5 lbs. of tea and 4 lbs. of sugar for 15s. 2d. Find the price of tea and sugar per lb.

3. A and B working together can earn 40s. in 6 days, A and C can earn 54s. in 9 days, and B and C together can earn 80s. in 15 days. Find what each man can earn alone per day.

4. A certain number of sovereigns, shillings, and sixpences amount to £8, 6s. 6d. The amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences. Find the number of each coin.

5. Explain what is meant by a *prime* number. Resolve 132288 into its prime factors. Show that if a prime number, greater than 3, be increased or diminished by unity, one of the results is divisible by 6.

6. A and B ran a race which lasted 5 minutes. B had a start of 20 yards; but A ran 3 yards while B was running 2, and won by 30 yards. Find the length of the course.

7. Reduce to their lowest terms the fractions $\frac{165}{45045}$ and $\frac{1065}{38766}$. Find their difference, and also the quotient of the first divided by the second. Simplify

$$\frac{2\frac{1}{13} \text{ of } \frac{5}{9} - 1\frac{1}{8} \text{ of } \frac{2}{3}}{1\frac{1}{21} \div 5\frac{4}{7}}.$$

8. Assuming that if 66420666 be divided by 7358, the quotient is 9027; write down the quotient—

(1) of 6642·0666 divided by 7358000;

(2) of ·066420666 divided by ·007358.

Simplify

$$\frac{\cdot 875 \times \cdot\dot{2}7\dot{0}}{\cdot 125 + \cdot 125\dot{6}7\dot{5}} + \frac{3}{53}.$$

9. Prove that the G. C. M. of two numbers is the L. C. M. of all their common measures.

10. A metre is defined to be worth $\frac{1}{10000000}$th of the one-fourth part of the circumference of the earth, and is equal to 39·37079 inches. Find the circumference of the earth in miles.

11. A debt of £65 is paid in francs valued at $9\frac{3}{4}$d. each, at a time when 25 francs are worth £1. What does the creditor gain or lose?

12. A decimetre is equivalent to 3·937 inches, and a cubic decimetre of water weighs one kilogramme. If a cubic inch of water weighs 252·45 grains, express a kilogramme in pounds avoirdupois, correct to within the one-thousandth part of a pound, 7000 grains being equal to one pound avoirdupois.

13. A person bought 659 eggs, some at 2 a penny, and others at 3 for twopence. He paid altogether £1, 9s. 11d. How many eggs did he buy at 3 for twopence?

14. Assuming that 3 hectares contain 35881 square yards, and that 1 hectare contains 10,000 square metres, find the length of a metre.

15. Assuming that an express train runs 40 miles an hour, and an ordinary train 30 miles an hour, and that the express fare is $\frac{1}{4}$d. a mile more than the ordinary, find how much an hour a man's time is worth, if it cost him the same to travel by the one as by the other.

16. Find a constant multiplier which will convert the number of lbs. of tea into their price in shillings, if 4 lbs. cost 9s.

17. Assuming that a cubic metre contains 1000 litres, and that a metre contains 39·4 inches, find the number of cubic inches in a litre.

18. A man spends every year one-tenth of his income, and invests the rest in annuities at the rate of £90 for every annuity of £3. Supposing his income £1000 a year to begin with, what will it be at the end of 4 years?

19. If 8 gold coins and 9 silver coins are worth as much as 6 gold coins and 19 silver ones, express a gold coin in terms of a silver coin.

20. The express train between London and Cambridge, which travels at the rate of 32 miles an hour, performs the

journey in $2\frac{1}{4}$ hours less than the parliamentary train, which travels at the rate of 14 miles an hour. Find the distance.

21. A person walked from Cambridge to a village at the rate of 4 miles an hour, and, on reaching the railway station, had to wait ten minutes for the train, which was then $4\frac{1}{2}$ miles off. On arriving at his rooms, which were a mile from the Cambridge station, he found that he had been out $3\frac{1}{4}$ hours. Find the distance of the village.

22. A person walked out a certain distance at the rate of $3\frac{1}{2}$ miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 5 minutes. He was out 25 minutes. How far did he run?

23. A man leaves his property, amounting to £7500, to be divided between his wife, his two sons, and his three daughters, as follows: a son is to have twice as much as a daughter, and the widow £500 more than all the five children together. Find how much each person obtained.

24. A body of troops retreating before the enemy, from which it is at a certain time 26 miles distant, marches 18 miles a day. The enemy pursues it at the rate of 23 miles a day, but is first a day later in starting, then after two days' march is forced to halt for one day to repair a bridge; and this they have to do again after two days' more marching. After how many days from the beginning of the retreat will the retreating force be overtaken?

25. A man buys a certain quantity of apples to divide among his children. To the eldest he gives half of the whole, all but 8 apples; to the second he gives half the remainder, all but 8 apples; in the same manner, also, does he treat the third and fourth child; to the fifth he gives the 20 apples which remain. Find how many he bought.

26. A waterman rows 30 miles and back in 12 hours, and he finds that he can row 5 miles with the stream in the same time as 3 against it. Find the time of rowing up and down.

27. A man bought a suit of clothes for £4, 7s. 6d. The trousers cost half as much again as the waistcoat, and the coat half as much again as the trousers and waistcoat together. Find the price of each garment.

28. Nine gallons are drawn from a cask full of wine, and it

is then filled up with water; then nine gallons of the mixture are drawn, and the cask is again filled up with water. If the quantity of wine now in the cask be to the quantity of water in it as 16 is to 9, find how much the cask holds.

29. A rows at the rate of $8\frac{1}{3}$ miles an hour. He leaves Cambridge at the same time that B leaves Ely. A spends 12 minutes in Ely, and is back in Cambridge 2 hours and 20 minutes after B gets there. B rows at the rate of $7\frac{1}{2}$ miles an hour; and there is no stream. Find the distance from Cambridge to Ely.

30. A person bought a certain number of eggs, half of them at 2 a penny, and half of them at 3 a penny. He sold them again at the rate of 5 for twopence, and lost a penny by the bargain. What was the number of eggs?

31. I divide a certain sum of money amongst A, B, C, D, E. I give A 8 less than $\frac{1}{2}$, and B 20 less than $\frac{1}{2}$ of this remainder, and C 10 less than $\frac{2}{3}$ of this remainder, and D 5 less than $\frac{1}{2}$ of this remainder, viz. £20, which is given to E. Find the sum.

32. Three men are employed in a work respectively 8, 9, and 10 hours per day, and receiving the same daily wage. After three days, each works 1 hour a day more, and the work is finished in three days more. If the total sum paid for wages be £2, 7s. $6\frac{1}{2}$d., how much of it should each receive?

33. A fruiterer bought 350 apples, some at 3 a penny, and the rest at 5 a penny. How many must he have bought at each price so that 2s. 8d. a hundred was $\frac{28}{25}$ of what he paid per hundred?

34. What fraction is £11, 12s. of £435? Reduce it to a fraction whose denominator is 100, and divide the resulting number by $3\frac{1}{5}$, leaving the denominator 100 as it was.

35. Find $\dfrac{\frac{75}{365} \times 4\frac{1}{3}}{100 + \frac{75}{365} \times 4\frac{1}{3}}$ of £472, 11s. 9d.

36. Give the answer and find the 100.

37. Give the answer and find £472, 11s. 9d.

38. Can you find the other data?

39. What is the worth of spirits per gallon, if adding 1 gallon of water for every 40 would make the diluted gallon worth 17s. 6d.?

40. Find what fraction of £34, 7s. 3d. is the difference between it and £34, 12s., and divide result by $\frac{63}{38500}$.

41. Find the alteration to be made in £34, 7s. 3d., so that the result of last answer may become exactly 4.

42. When the weight of 28 bushels of oats is $9\frac{3}{4}$ cwt., and that of 35 bushels of wheat is $17\frac{1}{4}$ cwt., what weight of oats is equal in bulk to 92 lbs. of wheat?

43. If B be $\frac{1}{3}$ of $5\frac{1}{4}$ of C, and D be $\frac{2}{7}$ of $8\frac{3}{4}$ of C, what fraction is B of D?

44. A and B rent a field for £60. A puts in 10 horses for $1\frac{1}{2}$ months, 30 oxen for 2 months, and 100 sheep for $3\frac{1}{4}$ months. B puts in 20 horses for 1 month, 40 oxen for $1\frac{1}{2}$ months, and 200 sheep for 4 months. If the food consumed in the same time by a horse and an ox is 3 times and twice that of a sheep, find the portion of the rent of the field which each must pay.

45. A courier is sent, who travels at the rate of 7 myriametres in 5 hours; 8 hours after his departure, another courier is sent to overtake him, who travels at the rate of 5 myriametres in 3 hours. In how many hours will the second courier overtake the first, and what number of myriametres will he have travelled?

46. A can do as much work in 6 hours as B can in 6, or as C can in 10. Supposing $\frac{1}{3}$ of a work to be done by A in 10 hours, and B to have afterwards worked at it for 14 hours, how long will it take C to finish it?

47. A page, who saved every year half as much again as he saved the previous year, had in seven years saved £102, 19s. How much did he save the first year?

48. A labourer has to spend 3s. $9\frac{1}{2}$d. a week on bread when it is $6\frac{1}{2}$d. the quartern. If it rises to 7d., how much less bread must his family eat in a week so that it may cost him the same as before?

49. In the making of pins, 3 men who file the points can keep at work exactly 5 men who put on the heads, and no man in either of these two sets can do the work of the other set. Suppose one of the first set to stay away for a week, by what fraction are the earnings of each of the remaining men diminished, supposing them to work by the piece and divide their earnings equally?

50. If 3 men or 4 women can execute a certain piece of work in $8\frac{1}{4}$ hours, in how many hours will 7 men and 9 women execute another piece of work 3 times as great, each person being able in the second case to do $1\frac{1}{5}$ of the work which he did in the same time in the first case?

Miscellaneous Questions.

I.

1. What number multiplied by 973 will give a product 368767?

2. How many times can 57 be subtracted from 10003?

3. Show that multiplication is a short way of doing addition.

4. What number divided by 393 will give a quotient 15792, with a remainder 313?

5. From what number must I take 8 more than its half to leave 30?

6. Find the largest number that will divide 3567 and 5547, and leave a remainder 3 each time.

7. Find the L. C. M. of $5\frac{1}{4}$d., 2s. $7\frac{1}{2}$d., 1s. $7\frac{1}{4}$d., and 10s. 1d.

8. In a mile race on bicycles A can give B 16 seconds' start. B can give C 352 yards, who is beaten by A in a minute. How long would each take to ride a mile?

II.

9. What number subtracted from the excess of a million over a thousand will leave $247689\frac{1}{2}$?

10. There is a number which is half as large again as a second number, and one-third as large again as a third, and the difference between the second and third is 12. Find the number.

11. From what number must I take away 7 less than two-sevenths, so as to leave 47?

12. Find the L. C. M. of $3\frac{1}{2}$, $4\frac{1}{2}$, $5\frac{1}{4}$, and $2\frac{4}{5}$.

13. Find the largest number that will divide 3791 with a remainder 17, and 5232 with a remainder 15.

14. Divide £379, 2s. 9d. amongst A, B, and C, so that A has £20 more than B, and £23 more than C.

15. Reduce $\frac{3\frac{1}{2}}{4\frac{1}{4}}$ to a fraction whose numerator is $4\frac{1}{4}$.

16. How much greater is $\frac{1}{5}$ of £2, 7s. 9d. than ·75 of 11s. 6d.?

III.

17. How much larger is the difference between $4\frac{1}{2}$ and $\frac{7}{8}$ than $\frac{4}{5}$ of $3\frac{1}{3}$ of $2\frac{7}{10}$ of $\frac{1}{9}$?

18. There is a number which is one-fourth as great again as a second which is half of 64. Find it.

19. Reduce the fractions $\frac{3}{5}$, $\frac{4}{7\frac{1}{2}}$, and $\frac{5}{8\frac{1}{4}}$ to fractions whose numerators are all 1, and state which of the 3 fractions is the least.

20. Find the G. C. M. of $14\frac{1}{7}$ and $15\frac{8}{9}$.

21. If A has $\frac{8}{7}$ of what B has, who has $\frac{4}{11}$ of what C has, who has $\frac{3}{11}$ of what D has, what fraction of A's is D's share?

22. If in the last question D had £21, 3s. 6d., what had A, B, and C?

23. Of what number is 18 greater by 2 than its third?

24. How many sevenths of a grain are there in the hundredth of a lb. troy?

IV.

25. How many times can I subtract $\frac{3}{7}$ from $5\frac{1}{4}$, and what is the remainder?

26. If I walk to a train, I am $\frac{1}{4}$ hour too late, but if I ride at 3 times the rate I am 5 minutes too soon. How far is it, if I can walk 3 miles in an hour?

27. From what number if I take away 29 will the remainder be double what it would be if I took away only 8?

28. What is the least sum of money which I could pay with coins worth 3s. $7\frac{1}{2}$d. or 4s. 9d. or 5s. $2\frac{1}{4}$d.?

29. Reduce 371324 nonary to the common scale.

30. Find the G. C. M. of 12997 and 14637.

31. A boy buys apples, some at 4 for 1d., some at 5, and some at 6, and sells them all at 5 a penny. If he bought equal numbers of each, why would he lose?

32. If he bought double as many at 5 a penny as at 4, how many must he buy at 6 a penny so as neither to win or lose?

V.

33. What number must I take away from 379, so that the quotient, after dividing the remainder by 67, is 2?

34. How many times is 2 oz. $3\frac{1}{3}$ drs. avoir. contained in 4 lb. 1 dr. avoir.?

35. Change 7 sq. poles 17 yds. into ninths of an inch.

36. What fraction can be subtracted from $4\frac{1}{2}$ seven times and leave a remainder $\frac{2}{3}$?

37. How many more times does twenty-seven contain five than thirty-five seven, and what is your remainder?

38. Evaluate $10\frac{3}{5}$s. + £$\frac{2}{7}$ − $\frac{4}{11}$ guinea, and reduce the result to a fraction of £2, 1s.

39. What number must I add on to 1008, so that it will contain 49 three times as often as it does with a remainder only half what it was before?

40. Find the number which is greater by 24 than its own fifth.

VI.

41. What is the least number of 6 figures that can be divided without remainder by 317?

42. Find the number from which 377 has been subtracted 17 times, so as to leave a remainder which contains 47 exactly 113 times.

43. In the following question, what unit could you take so as to avoid fractions? A stake has $\frac{1}{2}$ of its length in the mud, $\frac{1}{3}$ in the water, and 10 feet above.

44. How often does 4·7 contain ·033, and what is the denomination of your remainder?

45. In a 5 mile race A can give B 220 yds., who can give C 660 yds., who can only give D 100 yds. What ought A to give D?

46. If A only gave D 400 yds., and beat him by 1 minute, find how long he took to run 1 mile.

47. If 3 dollars are worth 13 shillings, and 4 shillings worth 5 francs, reduce 18200 francs to dollars.

48. Simplify the expression $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\dfrac{1}{5}}}}$ and compare it with the following fractions, $\dfrac{1}{2}$, $\dfrac{1}{2+\dfrac{1}{3}}$, $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4}}}$.

VII.

49. Find the sum of which 3s. 3d. is $\frac{2}{9}$, and reduce it to the fraction of £1, 3s. 3d.

50. Find, only using one aliquot part, the value of 2 qrs. 11 lbs. 3 oz. $3\frac{1}{5}$ drs. at £4, 7s. 6d. the quarter.

51. How many whole times is ·37 contained in 17, and complete the division to 5 places of decimals?

52. If the decimal equivalent to $\frac{4}{7}$ is $\cdot\dot{5}7142\dot{8}$, write down that equal to $\frac{1}{14}$.

53. 8 horses with 3 feeds a day can be kept 7 weeks for 2 guineas. How much will it cost to keep three times as many horses half as many weeks with 4 feeds a day?

54. Reduce $2\frac{2}{5}$ of £4, 7s. 9d. to the decimal of £5.

55. What sums of money can be reduced to a terminating decimal of £3?

56. Compare (1) as to which is the greatest and least, by reducing the numerators to 1 the fractions $\dfrac{3}{4\frac{1}{4}}$, $\dfrac{5\frac{1}{2}}{6\frac{4}{5}}$, $\dfrac{3\frac{1}{2}}{5}$; and (2) as to how much the middle one is less than the greatest.

VIII.

57. What fraction of 1 qr. is 6 lb. 11 oz. $8\frac{8}{25}$ drs.? Hence

58. Find, only using 2 aliquot parts, the value of 3 qrs. 21 lbs. 4 oz. $7\frac{17}{25}$ drs. at £3, 6s. 8d. per cwt.

59. How many times is $\frac{11}{14}$ grains contained in 3 drs. (avoir.)?

60. Multiply 3·14 (quinary) by 21·3 (quinary), and explain the nature of the fractional part.

61. How many times is 2·3 quaternary contained in 32·312, and of what denomination is your remainder?

62. If I take from a cask first 1 gallon less than half, and then 1 gallon less than one-third of what remains, and lastly,

1 gallon less than one-fourth of what remains, and find that I have still $12\frac{1}{4}$ gallons left, what did it contain at first?

63. Reduce the fraction $\frac{4s.\ 7\frac{1}{2}d.}{6s.\ 11\frac{1}{4}d.}$ to another fraction whose numerator is 11 gallons.

64. Find the fraction which is greater by $\frac{3}{8}$ than its own $\frac{117}{118}$.

IX.

65. Divide $1 \cdot 11$ by $\cdot 0037$. Hence write down the value of the following fractions as decimals, $\frac{11 \cdot 1}{\cdot 037}$, $\frac{1 \cdot 11}{37000}$, and $\frac{11100}{3 \cdot 7}$.

66. Subtract $\cdot\dot{5}7142\dot{8}$ of a guinea from £69, and reduce the result to the decimal of £5.

67. Multiply together the three sums capable of being formed with 37, 43, 53, by taking them two together.

68. I subtract from 500 a certain number; this remainder I multiply by $\frac{2}{5}$, and from this product I subtract 20 more than its quarter and have 100 left. What is the number which I subtracted from 500?

69. There are 3 quantities, one of which I wish to multiply by the quotient of the other two. If these quantities be (1) 4s., £5, and 6s. 8d., and (2) 4 lbs. 3 cwt. and 10 miles, in how many ways can I respectively perform the operation, and with what results?

70. A is walking from P to Q at $4\frac{1}{2}$ miles an hour, but on arriving ten miles from Q he meets B, with whom he turns and walks 2 miles at B's rate; if now he has to increase his pace $1\frac{1}{2}$ miles an hour so as not to be late, how fast does B walk?

71. If B arrives at P as A arrives at Q, what is the distance between P and Q?

72. A garrison provisioned for 2000 men for 65 days, receives after 10 days 1000 women and 1000 children; the men's rations are now reduced $\frac{1}{3}$, and 4 women or 7 children are served with what will suffice to keep alive 3 men. How long will the garrison hold out?

X.

73. Write down the smallest number in the quinary scale that will divide by the numbers 2, 3, 4, 6, 7.

74. Divide in the senary scale $43 \cdot 2$ by $1 \cdot 3$. Prove by reducing them to denary, and dividing and reducing back to the quinary.

75. Find price of $375\frac{3}{8}$ articles at £1, 16s. $2\frac{1}{2}$d. each.

76. A landlord, having a farm of 102 acres, previously let at £2 an acre, thrown on his hands, lets it out in 3-rood lots at 15 shillings a rood, and a bushel of potatoes for every lot. If he increases his income by £136, what is the value of potatoes per bushel?

77. Find, without dividing, the remainder, after dividing 4179 by 7, showing how you obtain it; also find it after dividing by 12.

78. If 8 men and 11 boys can accomplish as much work as 9 men and 8 boys, compare the powers of a man and a boy.

79. Find a number whose $\frac{1}{6}$ part is 7 greater than its $\frac{1}{7}$.

80. Change $3 \cdot 72$ into a radix fraction of which the radix is 9.

XI.

81. Find the remainder, without dividing, after 3731 has been divided by $3 \times 4 \times 5$.

82. A sells a horse to B and loses $\frac{1}{6}$ of what he gave for it. B sells it to C and gains $\frac{1}{7}$ of what he gave for it; C sells it to D at a gain of $\frac{1}{7}$ of what he gave for it, which was £400. What did the horse cost A?

83. If the owner of $\frac{3}{8}$ of a ship sell $\frac{4}{5}$ of $\frac{15}{17}$ of his share for £$\frac{3000}{175}$, what is the value of $\frac{2}{3}$ of $\frac{3}{7}$ of it?

84. Multiply in the quickest way possible 3712 by 625.

85. Multiply, only having 3 lines, 47123 by 12181.

86. How much is $\frac{1}{2}$ of 37 greater than $\frac{1}{3}$ of 55, and less than $\frac{1}{3}$ of 56?

87. Find the two smallest numbers whose halves differ by 3, and thirds by 2, the halves and thirds both being integers.

88. Divide £20,000 amongst a widow, 3 sons, and 5 daughters, giving the widow £1000 more than all the rest put together, the eldest son £1000 more than each of his brothers, and the eldest daughter £500 more than each of

her sisters, so that the youngest son and the eldest daughter have the same.

XII.

89. What is the difference between $\frac{7}{107}$ of £100 and $\frac{7}{100}$ of £100? and prove that this difference is equal to $\frac{7}{100}$ of $\frac{7}{107}$ of £100.

90. Find a number of which 317 is 11 greater than $\frac{1}{3}$ of its double.

91. $\frac{1}{7}$ of a barrel of brandy leaked away and was filled with water, after which $\frac{1}{11}$ was found to have leaked; when it was drawn off, and the water separated from the brandy, there were found only 29 gallons of pure brandy. What were the contents of the cask?

92. Find the G.C.M. of $\frac{712}{802}$ in the nonary scale.

93. How can you tell whether a number in the quinary scale is even?

94. Prove that 462 (septenary) is divisible by 6 and 8.

95. Find the number which is the same multiple of 5 that it is the measure of 605.

96. A and B have the same money in their pockets when they sit down to play chess. A wins 10s., and now he has three times as much as B. What had each at first?

PART II.

CHAPTER XI.

Ratio and Simple Proportion of Abstract Numbers—The Difference between Direct and Inverse Proportion.

1. The ratio between two numbers is a comparison as to how many times the one contains the other.

2. The measure of a ratio is the number of times the first contains the second; thus the measure of the ratio 8 to 4 is 2, that of 5 to 4, $\frac{5}{4}$.

3. If one of the terms of a ratio be a concrete quantity, it is axiomatic that the other quantity must be of the same denomination, or capable of being changed into the same denomination.

4. The measure of a ratio is always abstract. Hence the measure of a ratio between the terms of one denomination may be equal to the ratio between the terms of another denomination, and their measures may be multipled together or compounded. Thus the ratio between £1 and 6s. 8d. is $\frac{1}{3}$, that between 9 men and 7 men is $\frac{9}{7}$, and these measures, $\frac{1}{3}$ and $\frac{9}{7}$, compounded give us $\frac{3}{7}$.

5. When the measure of one ratio is equal to the measure of another ratio, the four quantities are said to be in proportion. Thus the measures of the ratio 5 to 7 and 10 to 14 are both of them $\frac{5}{7}$, therefore the four quantities 5, 7, 10, and 14 are said to be proportional. Since the measure of 5 to 7 is $\frac{5}{7}$, which is another way of writing down $5 \div 7$, this statement might be written down $5 \div 7 = 10 \div 14$.

The line in the symbol $\div$ is generally omitted and $::$ substituted for $=$ and the statement becomes $5 : 7 :: 10 : 14$. In this manual the sign $=$ will generally be written for $::$

6. If any 3 of these 4 terms are known, the remaining one can be found. Since $5:7=10:14$ can be written $\frac{5}{7}=\frac{10}{14}$. Now, if we multiply both of these expressions by the same number, we do not affect their equality; $\therefore$ let us multiply both fractions by the two denominators; thus—

$$\frac{5\times7\times14}{7}=\frac{10\times7\times14}{14},$$

or $5\times14=10\times7$.

Now, if we examine this last equality, we notice that the 5 and the 14, or the outside figures, multiplied together are equal to the 10 and 7, the inside figures, multiplied together.

Hence to find any term, if we have the inside figures, multiply them together and divide by the remaining figure, and the quotient will give us the unknown term. If we have the outside figures we must multiply them together, and divide the product by the remaining figure to find the unknown term; thus to find the second term in $3: \ =4:7$ we know the outside terms, viz. 3 and 7, which give a product of 21, and this divided by 4 gives $5\frac{1}{4}$ as our second term, hence our completed proportion is $3:5\frac{1}{4}=4:7$.

7. In Part I. we learnt how to reduce fractions to other fractions with a given denominator or numerator.

All questions in Simple Proportion are practically questions of this kind. In the foregoing question what we wanted to do was to reduce the fraction $\frac{4}{7}$ to one with a numerator 3, and to do this we first divide both numerator and denominator by 4, and then multiply both by 3; thus $\frac{4}{7}=\frac{1}{\frac{7}{4}}=\frac{3}{\frac{21}{4}}$, or $\frac{3}{5\frac{1}{4}}$, which gives us what we wanted.

8. Again, we want to find a number which is the same fraction of 3 that 7 is of 4. Now 7 is $\frac{7}{4}$ of 4, hence the number we want is $\frac{7}{4}$ of 3, or $5\frac{1}{4}$. The student is earnestly recommended to find wanting terms in proportions in every way, so as to get a complete grasp of the subject.

9. In the complete proportion

$$5:8=10:16,$$

if I double the 5 and leave the 8 and the 10 the same, I must halve the 16 if the four numbers are to remain in proportion; thus—

$$5 \times 2 : 8 = 10 : \tfrac{16}{2}.$$

But if I double the 5 and leave the 8 and the 16 unchanged, I must double the 10 if the four numbers are to remain in proportion; thus—

$$5 \times 2 : 8 = 10 \times 2 : 16.$$

Hence if two quantities are so connected together that an increase or decrease in the one produces a corresponding increase or decrease in the other, the first term must correspond with the third term, and the second with the fourth, for if I increase the first the third must be increased to retain the proportion. But if two quantities are so connected together that an increase in the one produces a corresponding decrease in the other, then the first and fourth must correspond, and the second and third, since if I increase the first I must decrease the fourth to keep the proportion. Now in the proportion

$$5 \text{ men} : 8 \text{ men} = £10 : £16,$$

if the question were one of wages according to the number of men employed; since an increase in the number of men is attended with a corresponding increase in the amount of wages, I know the 5 men earn the £10 and the 8 men earn the £16, but in the proportion

$$5 \text{ men} : 8 \text{ men} = 10 \text{ days} : 16 \text{ days},$$

if the question were one of time with respect to the number of men employed; since an increase in the number of men would involve a decrease in the time, I know the 5 men work in the 16 days, and the 8 men in the 10 days.

$$\text{Since } 3 : 5 = \tfrac{1}{10} : \tfrac{1}{16},$$

$$\text{or } \tfrac{3}{5} = \frac{\tfrac{1}{10}}{\tfrac{1}{16}} = \tfrac{16}{10},$$

$$\text{or } 3 : 5 = 16 : 10,$$

and $\frac{1}{10}$ is the reciprocal of 10,
and $\frac{1}{16}$,, ,, 16,

we can always change the places of the terms of a ratio by

writing their reciprocals; and remembering this, we can always make the first term correspond with the third and the second with the fourth by writing the reciprocals where the quantities are connected together inversely or indirectly, that is, when an increase in the one produces a corresponding decrease in the other. Thus, supposing we were asked to find the fourth when it was connected with 4 and the third (say 7) with 2 inversely, our proportion would be not

$$2 : 4 = 7 :$$

$$\text{but } \tfrac{1}{2} : \tfrac{1}{4} = 7 : \frac{7 \times 2}{4},$$

$$\text{or } 4 : 2 = 7 : \frac{7 \times 2}{4}.$$

10. In Algebra we prove many interesting properties of numbers and quantities in proportion, some of which are useful in arithmetic and can also be used in a legitimate manner.

$$\tfrac{5}{4} = \tfrac{10}{8};$$

let us subtract 1 from each of these equals—

$$\tfrac{5}{4} - 1 = \tfrac{10}{8} - 1,$$

$$\frac{5-4}{4} = \frac{10-8}{8},$$

$$\text{and } \frac{4}{5-4} = \frac{8}{10-8};$$

$$\text{and since } \tfrac{5}{4} = \tfrac{10}{8},$$

$$\therefore \tfrac{4}{5} = \tfrac{8}{10},$$

$$\therefore \frac{5-4}{4} \times \frac{4}{5} = \frac{10 \times 8}{8} \times \frac{8}{10},$$

$$\text{or } \frac{5-4}{5} = \frac{10 \times 8}{10},$$

$$\text{or } \frac{5}{5-4} = \frac{10}{10-8},$$

hence we can replace either of the two terms of the first ratio by their difference, and put either term for the other term, provided we do exactly the same with the terms of the other ratio. Similarly we might have added one and got the same rule with the word sum for difference. By and by we shall require to find the 6 (say) in the fraction $\frac{6}{100+6}$, so that the fraction may be equal to another fraction. The theorem we have just explained, if not proved, supplies us with a method of solving such questions. Supposing we are asked to find a fraction $= \frac{13}{20}$ whose numerator and denominator differ by 2, this really comes to reducing one of the fractions $\frac{13 \text{ or } 20}{20-13}$ to one whose denominator is 2. We say denominator, because the denominator must be greater than the numerator, since 20 is greater than 13. After the subtraction, then the 2 will be found in the denominator.

Let us reduce $\frac{13}{7}$ to a fraction whose denominator is 2.

$$\frac{13}{7} = \frac{\frac{13}{7}}{1} = \frac{\frac{26}{7}}{2};$$

$$\therefore \frac{13}{20} = \frac{\frac{26}{7}}{\frac{26}{7}+2}, \text{ the fraction required.}$$

We must not, of course, reduce this fraction, for if we did its terms would differ by 6, and not by 2, as they do now.

Let us work another example. Find a fraction equal to $\frac{2}{1\frac{1}{2}}$ whose terms differ by 100.

Here we must subtract the $1\frac{1}{2}$ from the 2 for the numerator and take the 2 (as the simpler) for the denominator; so our problem becomes, find a fraction with a numerator $100 = \frac{\frac{1}{2}}{2}$.

$$\frac{\frac{1}{2}}{2}=\frac{1}{4}=\frac{100}{400};$$

$$\therefore \frac{2}{2-\frac{1}{2}}=\frac{400}{400-100},$$

$$\text{or } \frac{2}{1\frac{1}{2}}=\frac{400}{300}, \text{ the fraction required.}$$

We have purposely omitted any rule for doing these questions, and strongly recommend the student to go through them step by step as we have done. If a rule were given, it is very doubtful whether one student out of ten would be able to follow its reasoning, and he would, moreover, be very apt to fall into mistakes.

Examination and Examples.

1. Distinguish between a ratio and its measure.
2. In compounding ratios, what do you really compound?
3. Compare the ratios $3 : 5$, $4 : 7$.
4. Compound the ratios 4s. : 5d. and 3 oz. : 12 lbs. avoir.
5. Can there be a ratio between 1 oz. avoirdupois and 1 oz. troy?
6. Fill up the terms wanting in the following proportions—

$$\quad : 2\tfrac{1}{2} = 3\tfrac{1}{4} : 2\tfrac{1}{7}.$$

7. $3\frac{1}{3} : \quad = 4 : 5\frac{1}{3}$.
8. $4\frac{1}{4} : 3\frac{1}{3} = \quad : \frac{3}{17}$.
9. $5\frac{1}{2} : 8 = 3\frac{1}{7} : \quad$.
10. Distinguish carefully between direct and inverse proportion.
11. What number bears the same proportion to 6 that 7 does to 8?
12. Place the four numbers 6, 2, 3, 4 so that they form a proportion in as many ways as you can.
13. Find a number which 7 contains as often as 4 contains 5.

14. What number is contained in 6 as often as 5 is contained in 7?

15. Find the ratio of which the first term is 3 whose measure is equal to that of the ratio between 7s. and 4s. 1d.

16. Form a ratio of which the second term is 6 whose measure is 3 times that of the measure of 2 lbs. to 3 lbs. 7 oz.

17. If the number of men, and time that they do their work in, form an inverse proportion, what is expressed by the following statement:—

3 men : 5 men = 6 weeks : 10 weeks?

18. When are quantities connected together directly and when inversely?

19. Find the fraction which contains $2\frac{1}{2}$ as often as $\frac{3}{5}$ contains $\frac{1}{7}$.

20. Fill in the terms wanting in the following proportions:—

1·2 : 3·4 = 5·6 .

21.   : 1·2 = 3·4 : 5·6.

22. 1·2 :   = 3·4 : 5·6.

23. $1\cdot\dot{1} : 2\cdot\dot{3} = 3\cdot\dot{4} \quad\quad .$

24. $\dfrac{1\cdot\dot{2}}{2\frac{1}{2}} : \quad\quad = \dfrac{2\cdot\dot{4}}{15} : \dfrac{17\frac{1}{3}}{5}.$

25. Multiply £3, 6s. 8d. by 3s. : 10s.

26. Multiply 4 oz. 2 drs. by 3 men : 11 men.

27. Divide 5 cwts. 1 qr. by £7 : £3.

28. Divide £2, 3s. 4d. by $\frac{1}{3} : \frac{1}{18}$.

29. What is the difference between the measures of ·03 : ·005 and $\cdot0\dot{3} : \cdot0\dot{0}\dot{5}$?

30. What is the difference between the measures of 2 lbs. : 7 lbs. and 4 men : 9 men?

31. Find a ratio of men whose measure is greater by $\frac{3}{5}$ than that between 4 horses and 7 horses.

32. Find a ratio between pounds, etc., troy, whose measure is less by $\frac{2}{7}$ than that between 4 men and 11 men.

33. Find a ratio between shillings whose measure is $\frac{7}{5}$ of that between 4 lbs. and 11 lbs. 6 oz. (avoir.).

34. Find a fraction whose terms differ by 2 equal to the fraction $\frac{8}{7}$.

35. Find a fraction whose terms differ by 3 equal to the fraction $\frac{10}{9}$.

36. Find a fraction whose terms differ by 100 = the fraction
$$\frac{£2,\ 10\text{s.}}{£44,\ 3\text{s. }4\text{d.}}$$

37. Find a fraction whose terms differ by 6 = the fraction
$$\frac{£41,\ 13\text{s. }4\text{d.}}{£44,\ 3\text{s. }4\text{d.}}$$

38. Find a fraction whose terms differ by $\frac{1}{3}$ equal to $\frac{2}{3}$.

39. Of what fraction is the numerator 10 greater than the denominator, if its value be $\frac{3}{2}$?

40. Find the fourth term in

$$\frac{5\frac{1}{2}}{2\dfrac{1}{1\frac{1}{2}}} : \frac{2}{3\dfrac{2}{5\frac{1}{2}}} = \frac{3\frac{1}{4}}{\frac{2}{37}} :$$

The remaining 10 questions will be given in Chapter XII.

CHAPTER XII.

Questions in Simple Proportion worked out—(1) by the Unitary Method; (2) by Fractions; (3) by the Completing a Fraction; (4) by the Adoption of a Unit; and (5) by the Method of Proportion or Rule of Three.

1. Some few years ago a raid was made against the old method of Rule of Three, on the idea that it implied an absurdity to multiply one concrete quantity by another.

In answering the simplest question about quantities, we have to do as much as we have in the old Rule of Three, *e.g.* supposing I ask how many legs have 28 donkeys, we seem to multiply the 28 donkeys by the 4 legs, and get a result 112 legs. Of course what we really do is to take the 4 legs 28 times, because the 4 legs are repeated as often as there are donkeys.

2. Now supposing we have to find the price of 6 lbs. if 8 shillings will buy 4 lbs., it is evident that the ratio of the weight is equal to that of the price, and that the proportion is direct, as an increase in the weight is attended by a corresponding increase in the price. It does not matter which of the four terms we take as the unknown quantity. In this country we usually leave the last as the unknown term. In some countries children are taught to leave the first as the unknown term. Let us take the last. This term being shillings, the third term must be money. The proportion being direct, the third term corresponds to the first—that is, pays for the first weight, and the second weight will be paid for by the second sum of money. So we have—

4 lbs. : 6 lbs. = 8s. : unknown number of shillings.

Now the 8 is to contain the unknown number of shillings as often as the 4 lbs. contain the 6 lbs., or, what is the same thing, the unknown quantity contains the 8s. as often as 6 lbs. contain 4 lbs.

Now 6 lbs. contain 4 lbs. $\frac{6}{4}$ or $\frac{3}{2}$ of a time, and therefore the 8s. must be taken $\frac{3}{2}$ of a time, which gives us 12s.

When, therefore, we multiply the 8s. by 6, we no more multiply the shillings by lbs. than we do in the question of the donkeys and their legs.

3. On account of this misunderstanding, the following method, called the unitary method, was introduced, and is now required by the Educational Code:—

If 4 lbs. cost 8s.,
1 lb. costs $\frac{8}{4}$ or 2s.;
$\therefore$ 6 lbs. cost 12s.

This no doubt is very simple, but in some cases not so easy to explain. For instance, if 317 children out of 390 pass an examination, how many will pass out of 1000?

If 390 children supply 317 passes,
$\therefore$ 1 child supplies $\frac{317}{390}$ pass,

$$\text{and 1000 children supply } \frac{317 \times 1000}{390} \text{ passes,}$$

or 812·820, etc.;

but I doubt if many children or their teachers would grasp the idea of the $\frac{317}{390}$ of a pass.

4. Another method may be called the fractional method, in which we multiply the single term by the fraction formed by the terms of the other or complete ratio thus. To take the same question as we worked out above about the children's passing:

Our complete ratio is between the number of candidates, viz. 390 and 1000, and our incomplete ratio is one of passed children, of which we only know the term 317.

Now 1000 candidates will give more passes than 390, hence the fraction by which we multiply 317 must be greater than 1 —that is, we must multiply the 317 by $\frac{1000 \text{ candidates}}{390 \text{ candidates}}$, and not $\frac{390 \text{ candidates}}{1000 \text{ candidates}}$. This method is practically useful, but not very satisfactory as a mental exercise.

5. Again, we may solve the question by the method of completing a fraction, of which we know one term, to make it *similar* to another. The number of candidates form a fraction

$\frac{390}{1000}$, and we have to form a fraction whose numerator or denominator is 317. Since the proportion is direct, we know that the numerators correspond and the denominators—that is, that the candidates whose number is given in the numerator will provide the passes found in the numerator, and those found in the denominator will supply the others. Hence our question is: Find the fraction whose numerator is 317, equal to $\frac{390}{1000}$, and the operation is, as shown in Part I.—

$$\frac{390}{1000} = \frac{1}{\frac{1000}{390}} = \frac{317}{\frac{1000}{390} \times 317} = \frac{317}{812\frac{32}{39}}.$$

6. Often by the adoption of a proper unit we can reduce most complicated problems in Proportion, both Simple and Compound, to very easy questions, in which nothing but a knowledge of the four rules is required. We will take another example, which will illustrate this better than the one about the children. If 3 men can perform a piece of work in 8 days, in how many days would the work be done by 6 men?

If we take a man's day's work as our unit, it is evident that there are 24 units of work to be done, which 6 men could do in 4 days. Hence 4 days is our answer.

7. Finally, there is the old method of Rule of Three, in which I would urge students to place the terms of the complete ratio by the test of the quantities being connected together directly or inversely, and not by that as to whether the answer will be larger or smaller than the third term.

8. We will now work out in these five ways the following question:—If 8 horses will consume the grass of a field in 12 weeks, in how many weeks will 16 horses consume it?

(1) *The Unitary Method.*

8 horses consume the grass in 12 weeks,
1 horse consumes it in 12×8 weeks,
16 horses consume it in $\frac{12 \times 8}{16}$ weeks.

Ans. 6 weeks.

(2) *The Fractional Method.*

The answer must be less. We must therefore write the terms of the complete ratio so that its measure is less than 1, thus: $\frac{8}{16}$;

$\therefore$ number of weeks is 12 weeks $\times \frac{8}{16}$, or 6 weeks as before.

(3) *Completion of a Fraction Method.*

Our complete ratio is $\frac{8}{16}$, and our proportion is indirect; therefore the 12 weeks which correspond to the 8 horses, which we find in the numerator, must be in the denominator, and we have to find the numerator of the equivalent fraction whose denominator is 12. Thus—

$$\frac{8}{16} = \frac{\frac{8}{16}}{1} = \frac{\frac{8}{16} \times 12 \text{ weeks}}{12 \text{ weeks}} = \frac{6 \text{ weeks}}{12 \text{ weeks}}.$$

Hence again our answer is 6 weeks.

(4) *Adoption of a Unit.*

Our unit here is evidently the amount of grass a horse eats each week.

8 horses in 12 weeks eats 96 units;

$\therefore$ 16 horses will eat these 96 units in 6 weeks.

(5) *Rule of Three.*

Our third term is weeks. The proportion is indirect; therefore the horses which correspond to the 12 weeks (the former term in the second ratio) must be the latter term of the complete ratio. It is well for a time to write to the left the letter D for direct and I for inverse, and to connect the corresponding terms as done below.

I 16 horses : 8 horses = 12 weeks : Ans.

We could, of course, have made the former term of one ratio

correspond with the former term of the other by writing the reciprocals of the terms of the complete ratio, thus—

$$\frac{1}{8} : \frac{1}{16} = 12 \text{ weeks} : \text{Ans.}$$

$$= \frac{12 \times 8}{16} = 6 \text{ weeks.}$$

If we make the first term the unknown term, and place the terms according to the plan taught in this fifth method, we get practically the second method, thus—

unknown weeks : 12 weeks = 8 horses : 16 horses,

in which we are asked what number of weeks contains 12 weeks as often as 8 contains 16, to answer which we multiply the 12 by $\frac{8}{16}$, as in the second method.

9. If a term is common to both conditions, it does not affect the answer, and can be ignored. Thus supposing we had stated that these weeks were weeks of 7 days, as a feeding week would naturally be, the 7 would not appear at all in the working.

10. When all the terms are of the same denomination, as we found children in the question worked out above, we must separate them by some other name. For instance, if £500 will earn £12, how much will £14 earn? Now it is evident here that the £500 and the £14 are the moneys that earn money, and the £12 is a sum of money that is earned. The proportion is direct, so the statement is—

£500 : £14 = £12 : earnings required.

$$£\frac{14 \times 12}{500} = 6\text{s. } 8\tfrac{16}{25}\text{d.};$$

or by the method of units,

$$£500 \text{ earn } £12,$$

$$£1 \text{ earns } £\frac{12}{500},$$

$$£14 \text{ earn } £\frac{12 \times 14}{500} = 6\text{s. } 8\tfrac{16}{25}\text{d.}$$

11. The student is recommended to work out a great many examples in all five methods suggested here, and he will be surprised what a grasp he will get of the entire subject.

12. When additions and subtractions occur in the working of any questions, it is always as well, if possible, to work the question by means of a unit.

13. As a rule, it will be found easier to find a unit in questions of inverse proportion than of direct proportion, and in some cases, especially when the terms are all the same, it is often difficult to find an intelligible unit.

14. Sometimes the term of the proportion that we find is not the thing asked for, though we generally can make it so by performing some operation upon the quantities before we introduce them into the statement.

Supposing I ask what will a man lose if he only receive 5s. 6d. in the £ on a claim of £200?

If I state this thus—

debt. debt. receipt. receipt.

£1 : £200 = 5s. 6d. :

I find his receipts £55.

Hence to get his loss, we must subtract this from £200. But if we insert his loss on £1, viz. 14s. 6d., as our third term, then our incomplete ratio will be one between losses, and the fourth term, when found, will be the loss asked for. Thus—

debt. debt. loss. loss.

£1 : £200 = 14s. 6d. :

which gives at once the answer required, £145.

Examination and Examples.

1. How much coffee at 1s. $10\frac{1}{2}$d. per lb. must be given in exchange for 78 lbs. of tea at 5s. $1\frac{1}{2}$d.?

2. If 1000 sovereigns weigh 21 lbs. 5 oz. 16 dwts., what weight of gold will be contained in 192 sovereigns?

3. If 152 sacks will hold 65 qrs. 2 bush. 2 pks., how many sacks will contain 18 qrs. 7 bush. 1 pk.?

4. If 2 ro. 15 po. cost £59, 7s. 6d., what will 8 ac. and 17 po. cost?

5. If 17 yds. of silk cost £4, 8s. 6½d., what will be the cost of 120 yds. at the same rate?

6. How many yards of cloth worth 18s. 3d. per yd. ought to be given in exchange for 24 English ells of cloth worth 13s. 8¼d. per ell?

7. How long will 12 men take to do a piece of work which 8 men can do in 27 days?

8. What is the price of beef per lb. when 4 cwts. may be bought for 16 guineas?

9. If 12 bushels of wheat cost £4, 1s. 6d., how much can be bought for £55, 0s. 3d.?

10. If three men can mow a field of 15 acres in 6⅔ days, how long will it take 10 men to mow the same field?

11. If 45 yds. of silk be bought for £29, 18s. 3d., how many will cost £9, 19s. 5d.?

12. If the railway fare for a journey of 50 miles be 12s. 6d., what will be the fare for a journey of 120 miles?

13. If 17 men can earn £6, 7s. 6d. in two days, how many men will earn £5, 12s. 6d. at the same rate in the same time?

14. The cost of keeping 25 horses being at the rate of £11, 6s. 0½d. per week, what will be the cost of keeping 13 horses from September 1 to October 31?

15. How many men in 19 days would do a piece of work which 171 men could do in 12 days?

16. How far can 48 tons be carried for the money paid for carrying 36 tons 144 miles?

17. A coach travels 7½ miles an hour; how far will it have gone between 10.15 a.m. and 5.45 p.m.?

18. When an income of £1150 is reduced by taxation to £1063, 15s., what is the income of a person who has £518 left after deducting the taxes?

19. If 3 men do a piece of work in 60 days, how many will do it in 20 days?

20. The wages of 36 men amount to £28, 16s.; what is the amount of the wages of 60 men for the same time at the same rate?

21. Supposing each penny in the £ of the income tax to yield a million sterling to the revenue, find the assessable income of the country.

22. How much will a poor-rate of 2s. 8d. in the £ produce in a parish where the whole property is rated at £4736, 5s.?

23. If a man can row 3 miles an hour against a stream, the rate of which is $2\frac{1}{2}$ miles per hour, how far will he row in $3\frac{1}{2}$ hours with the stream?

24. The poor-rate being 3s. 4d. in the £, what is the rateable value per annum of a property which pays annually a rate of £416, 13s. 4d.?

25. If the price of 3000 copies of a book be £4725, what sum will the sale of 1937 copies produce?

26. If 18 gallons of beer cost £1, 1s., what is the cost of 11 pints?

27. If 18 yds. of cloth cost £15, 10s. 6d., how much will 11 feet cost?

28. If a man can walk 11 yds. in 5 seconds, how many miles can he walk in an hour at the same rate?

29. If a carrier pigeon flies $17\frac{1}{2}$ miles in 15 minutes, how long will it take to fly 14 miles?

30. If a field containing 5 ac. 1 rd. produce 28 qrs. of wheat, what is the area of a field which produces 100 qrs.?

31. If 57 workmen can make an excavation in 9 days, how many can do it in three times as long?

32. If £5, 10s. be the cost of 11 yds. of cloth, how many pieces, each containing 9 yds., could be purchased for £135?

33. If I lose 5s. on an article which cost me £2, what do I lose on one which cost me 8s. 4d.?

34. If I gain 8s. on an article which cost me £3, what do I gain on one which cost £5?

35. Find in miles the rate per hour if I travel 2 miles 4 fur. 20 po. in 3 min. 15 sec.

36. If £100 earn £4, 10s. in a year, how much will £750 earn in the same time?

37. If £100 earn £5 in a year, in what time would £80 earn the same?

38. How long will A take to do on a bicycle at 15 miles an hour what B can walk at 4 miles in 2 hrs. 15 min.?

39. A starts on a bicycle 2 hrs. 15 min. after B has started, walking at $3\frac{1}{2}$ miles an hour, and catches him up after B has walked 12 miles. Find A's rate per hour.

40. The weight of 22 sovereigns is 5 oz. 13 dwts. How many sovereigns are coined from $28\frac{1}{4}$ lbs. of standard gold?

41. If eggs bought at $11\frac{1}{2}$d. a dozen are retailed at 9s. 7d. a hundred; what do I receive for eggs which cost me 100d.?

42. If for eggs which cost me 100d. I receive $104\frac{1}{6}$d., what shall I receive per dozen for those which cost me 9d. for 10?

43. If a salesman only receives $91\frac{7}{8}$d. for that which cost him 100d., what did he lose on a fowl which he sold for 2s. 11d.?

44. How many gallons of spirits at 14s. $10\frac{1}{2}$d. must I sell so as to receive the same as if I sold $31\frac{1}{2}$ gallons at 15s. 7d.? Hence—

45. How much water must be added to $31\frac{1}{2}$ gallons of spirits so as to reduce its price from 15s. 7d. to 14s. $10\frac{1}{2}$d.?

46. A bankrupt's debts are £1680 and his assets £790; how much did the creditors get per £?

47. If the price of the 4 lb. loaf should be 6d. when wheat is at 50s., what should it be when wheat is at 60s. a quarter?

48. If the sixpenny loaf weighs 3 lbs. 8 oz. when wheat is at 50s. a quarter, what ought it to weigh when wheat is at 60s.?

49. I bought 35 gallons of spirits at 17s. 6d. a gallon, but after adding to it 14 gallons of water I only received 96s. for what cost me 100s. At what price did I sell the mixture per gallon?

50. If £25, 13s. is $\frac{108}{100}$ of the buying price, what fraction (with 100 for its denominator) of the buying price is £24?

51. Compare the price per lb. of mutton and salmon, when a leg and a shoulder of mutton weighing 12 lbs. and 8 lbs. respectively is equivalent to a salmon weighing 16 lbs. 4 oz. Is this a direct or an inverse proportion?

52. If 6 men of $\frac{1}{4}$ better than ordinary expertness can perform a piece of work in 16 days, in what time would the same number of men of $\frac{1}{3}$ less than the ordinary expertness do it?

53. With what quantity of spirits, worth 18s. a gallon, must $3\frac{1}{2}$ gallons of water be mixed to reduce the price per gallon to 16s.?

54. The rent of an orchard is paid by selling the apples at 12s. for a bushel and a half when the number of apples produced is 8442. At what price per quart must the apples be sold to pay the rent when the number produced is 7236?

55. Form a proportion in men and days, the terms of one of the ratios differing by 3, the product of the extremes being equal to 32.

56. If 6s. 4d. is $\frac{109\frac{1}{4}}{100}$ of the buying price, at what price must it be sold so that the selling price is $\frac{115}{100}$ of the buying price?

57. When 52 bushels of corn are worth £17, 11s. (of average quality), what is the value of the same number of one-third better than average quality?

58. If a person's gross income be £414, to what will it be reduced if he have to pay an income tax of 5d. in the £?

59. If my net income, after paying income tax, is £388, 12s., and my gross income £400, what is the amount of the income tax in the £?

60. If the difference of 1d. in the £ in the income tax alter my income by £25, 10s., what is my gross income?

CHAPTER XIII.

Percentages—Commissions—Insurance—Profit and Loss—Constant Multipliers.

1. To compare prices, gains, losses, etc., it is convenient to compare them with some fixed quantity.

2. Supposing A lost 1s. in every £, and B lost 1d. in every shilling, to compare these losses we must find what they respectively lose on some one and the same quantity.

3. The quantity usually used is 100 of that denomination to which either quantity can be most easily reduced. Thus, in this case let us take 100s. Then A loses 5s. on his 100s., and B loses 100d. or 8s. 4d. on his 100s. So that A is said to lose 5 per cent. or 5s. on his 100s., and B loses $8\frac{1}{3}$s. on his 100s.

4. All questions of per-centage can be reduced to questions in Proportion thus. If I buy goods for £8, and sell them at £9, what do I gain per cent.? may be asked thus: If for £8 I receive £9, what shall I receive for those which cost me £100? of which the statement is—

give. give. receive. receive.
£8 : £100 = £9 : £ to be found.

$$\text{Ans. } £\frac{100 \times 9}{8}, \text{ or } £112, \text{ 10s.}$$

This is, of course, a direct proportion.

5. Commission and insurance are almost always charged at so much per 100. Thus, in the question, What is the commission on £500 at 5s., or $\frac{1}{4}$ per cent.? the statement becomes (being direct)—

£100 : £500 = £$\frac{1}{4}$: £s to be found.

$$\text{Ans. } £\frac{500 \times 1}{100 \times 4} = £1, \text{ 5s.}$$

6. At what must I sell goods, which cost me £180, so as to make 10 per cent.? This really means, If for goods which cost me £100 I receive £110, what shall I receive for those which cost me £180?

$$\begin{array}{cccc} \text{cost.} & \text{cost.} & \text{receive.} & \text{receive.} \\ £100 : & £180 = & £110 : & \text{to be found.} \end{array}$$

$$£\frac{180 \times 110}{100} = £198.$$

By the method of fractions explained in Chap. I., to find this £198 I should have to multiply £110 by $\frac{180}{100}$; but to multiply £110 by $\frac{180}{100}$, since the order of multiplication is immaterial, is the same as multiplying £180 by $\frac{110}{100}$, and *practically* this method is the best. Similarly, What do I receive for goods which cost me £120, if I sell them at a loss of 8 per cent.? The best method is to multiply the £120 by $\frac{92}{100}$ (92 being $100 - 8$).

7. When all the four terms of a proportion are of the same kind, we can alternate the second and third terms. So

$$£4 : £5 = £6 : £7,\ 10s.$$

may be written—

$$£4 : £6 = £5 : £7,\ 10s.;$$

or, to take another example—

$$£4 : £2,\ 8s. = 5s. : 2s.\ 6d.$$

may be written—

$$£4 : 5s. = £2 : 2s.\ 6d.$$

8. What is the difference between a commission of 6d. in £5 and 2s. 6d. in the £100? 6d. in £5 is $\frac{1}{200}$, and 2s. 6d. in £100 is $\frac{1}{800}$, and their difference is $\frac{4-1}{800}$, or $\frac{3}{800}$, which might be written $\frac{\frac{3}{8}}{100}$, or $\frac{7s.\ 6d.}{£100}$; that is, a difference of 7s. 6d. in every £100.

9. If an income tax of 5d. in the £ is subtracted from an income, since 5d. in the £ is $\frac{5}{240}$ or $\frac{1}{48}$, $\frac{47}{48}$ of the income must be left. Hence the easiest way to find a net income is

to work by the fractional method. For example, What is the net income after 8d. in the £ has been subtracted from a gross income of £630? The result is immediately found by multiplying £630 by $\frac{240-8}{240}$, or $\frac{232}{240}$, or $\frac{58}{60}$, or $\frac{29}{30}$, which gives us £609.

10. By the principle of reciprocity, to find the gross from the net, we must multiply the net income by $\frac{30}{29}$.

11. If I buy eggs at 4s. 6d. a hundred, what is the difference in what I receive for them a dozen, if I sell them at a profit of 10 per cent. or 12 per cent.?

In the first case, for 100 eggs I receive $\frac{110}{100}$ of 4s. 6d.
" second " " $\frac{112}{100}$ "
∴ for a dozen in the first case I receive $\frac{110}{100}$ of 4s. 6d. $\times \frac{12}{100}$,
and " second " $\frac{112}{100}$ " "
and the difference between them is—

$$\frac{112-110}{100} \text{ of } 4\tfrac{1}{2} \times \tfrac{12}{100}\text{s.}$$

$$= \tfrac{1}{50} \times \tfrac{9}{2} \times \tfrac{12}{100}\text{s.} = \tfrac{81}{625}\text{d.}$$

12. This is easy; but to find one of the other elements if the difference were given is not by any means so easy. Let us find the 4s. 6d. The question would be given thus: At what price per hundred must I buy eggs so that by selling them at 12 p. c. profit I make $\frac{27}{625}$d. more on every dozen than if I had sold them at 10 p. c. (note p. c. and °/₀ both stand for per cent. or per centum) profit?

The difference on each dozen is $\frac{81}{625}$d., but the difference on each dozen at the two rates is $\frac{112-110}{100}$, or $\frac{1}{50}$ of the buying price of a dozen;

∴ $\frac{81}{625}$d. is $\frac{1}{50}$ of the buying price of a dozen;

∴ the buying price of a dozen is $50 \times \frac{81}{625}$d., and the buying price of 100 is $\frac{50 \times 81}{625} \times \frac{100}{12}$d., or 54d., or 4s. 6d.

13. All the other elements could be found. We will find one other, and leave the rest to the student. Let us find the larger profit, viz. 12 p. c. The question then is, I buy eggs

at 4s. 6d. a hundred, and sell them at a certain profit; but I find that if I sold them at 10 p. c. I should receive $\frac{81}{625}$d. less per dozen. What is the larger profit per cent.?

The working is like this—

$\frac{110}{100}$ of 54d. is the selling price of 100 at the smaller profit;
$\therefore$ $\frac{110}{100}$ of $54 \times \frac{12}{100}$d. is the selling price of a dozen at the smaller profit, and $(\frac{110}{100}$ of $54 \times \frac{12}{100} + \frac{81}{625})$d. is the selling price of a dozen at the larger profit, and $\frac{54 \times 12}{100}$d. is the buying price of a dozen;
$\therefore$ the following proportion will give us the selling price of what cost 100d.—

$$\begin{array}{llll} \text{buy.} & \text{buy.} & \text{sell.} & \text{sell.} \\ \frac{54 \times 12}{100}\text{d.} & : 100\text{d.} = & (\frac{110}{100} \text{ of } 54 \times \frac{12}{100} + \frac{81}{625})\text{d.} & : \text{pence required,} \end{array}$$

and this gives 112d.;
$\therefore$ answer is 12 per cent.

14. In paragraph 7 we said we could interchange the second and third terms, provided the ratios were of the same kind. If, however, we *abstract* the denominations, we can always, as far as practical results are concerned, interchange the middle terms, and so get a connection between the number which expresses one property of a body and that which connects another.

E.g. if 4 lbs. cost 7s., what will 3 lbs. cost? Here the statement is 4 lbs. : 3 lbs. = 7s. : s. required, which gives us 7s. $\times \frac{3}{4} = 5\frac{1}{4}$s., since $7 \times \frac{3}{4}$ is the same as $3 \times \frac{7}{4}$.

If I multiply the weight (expressed in lbs.) by the abstract number $\frac{7}{4}$, I immediately obtain the price expressed in shillings. If I want to find a constant multiplier in whole numbers, I must, of course, change one of the units. In this case, if we change the unit of money to threepenny pieces, we immediately obtain the price of a parcel by multiplying the number of lbs. by 7.

E.g. 7 lbs. will cost 49 threepenny pieces, 28 lbs. 196, etc.

If we retain shilling as our unit of price, we must change our unit of weight into 4 lbs. Thus, to find the price of 28 lbs., the answer is 7×7 or 49s., of 30 lbs. it is $7\frac{1}{2} \times 7 = 52\frac{1}{2}$s., etc.

15. It may be here mentioned that this is the method always adopted in Algebra in working out these questions. It will be referred to again and again as we explain other applications of Proportion.

EXAMINATION AND EXAMPLES.

1. What do you understand by the expression 5 per cent.? Give an illustration.

2. If an income of £300 pays £11, 5s. for income tax, how much will be paid for an income of £525, 10s.?

3. The rental of a parish is £1850, and the rates are taken on 80 per cent. of the rental. What is the amount of a rate at 9d. in the pound?

4. If $5\frac{1}{2}$ per cent. would be gained by selling 121 yards of silk for £26, 11s. $10\frac{3}{4}$d., at what price per yard must it be sold to gain 12 per cent.?

5. Find how much in the £ a bankrupt pays if his creditors receive £188, 2s. 9d. out of £6228.

6. A grocer mixes 72 lbs. of tea at 2s. $10\frac{1}{2}$d. per lb. with 90 lbs. of tea at 2s. 6d. per lb. At what price per lb. must he sell the mixture so as to gain 5d. per lb.?

7. The cost of 5 cwts. 2 qrs. 14 lbs. of coffee is £56, 13s. 9d. If the whole is sold at 1s. $11\frac{1}{4}$d. per lb., what is the total gain or loss?

8. A person selling an article for £2, 12s. 6d. gains 5 per cent. What would he have gained or lost per cent. by selling it at £2, 7s. 6d.?

9. A man bought 240 cows at Rotterdam, paying £14, 6s. for each; their carriage to London cost $\frac{1}{22}$ of his outlay; he then exchanged them for 299 ponies, which he sold for £15 per head. What did he gain per cent. on his expenditure?

10. A grocer buys 1 cwt. of sugar for £1, 13s. 4d., and sells it for £2, 2s. 6d. What is his gain per cent.?

11. If a farthing be the interest on a shilling for a calendar month, what is the rate per cent. for same time?

12. How much will a poor-rate of 2s. 8d. in the £ pro-

duce in a parish where the whole property is rated at £4736, 5s.?

13. A debt of £65 is paid in francs, valued at $9\frac{3}{4}$d. each, at a time when 25 francs are worth £1. What does the creditor gain or lose per cent.?

14. An Indian officer, whose annual pay was estimated in rupees, lost £41, 12s. 6d. in one year by a fall in the value of the rupee from 1s. $11\frac{1}{2}$d. to 1s. $10\frac{1}{4}$d. What was his salary, estimated in rupees?

15. One-twelfth of the weight of a gold coin is alloy of no value. If some sovereigns were coined of pure gold, how many of our present sovereigns would be equal in value to 143 of the new ones?

16. How much is gained or lost per cent. by buying a number of oranges at 5 for twopence, and selling half of them at 2 a penny and half at 3 a penny?

17. In a village of 1000 people it is found that 515 are females. Find the percentage of males.

18. Find the premium on a policy of assurance for £1258, 5s. at $6\frac{1}{4}$ per cent.

19. By selling a carriage for £73, 3s. I should lose 5 per cent. At what price must I sell to gain 15 per cent.?

20. If 20 per cent. be lost on a horse sold for £19, 4s., what was the cost of the horse?

21. A dealer buys 15 horses at £28, 13s. each, and sells 8 of them at £35 each and the remainder at £42, 10s. each. Find his total gain.

22. When an income of £1150 is reduced by taxation to £1063, 15s., what is the income of a person who has £518 left after deducting the taxes?

23. The poor-rate being 3s. 4d. in the £, what is the rateable value per annum of a property which pays annually a rate of £416, 13s 4d.?

24. A draper bought 600 yards of silk at 3s. $4\frac{1}{2}$d. per yard; and having sold 360 yards at 4s. 6d. per yard, and 81 yards at 3s. 9d. per yard, he was robbed of the rest. What was his whole gain or loss per cent. on his outlay?

25. A grocer buys $6\frac{1}{2}$ cwts. of tea at the rate of 17 guineas per cwt. He sells 3 cwts. of it at 3s. 3d. a lb. and the

remainder at 3s. $9\frac{1}{2}$d. a lb. How much per cent. does he gain?

26. A man bought 63 sheep, and sold $\frac{4}{9}$ of them at a profit of 15 per cent., $\frac{1}{7}$ at a profit of 50 per cent., and the remainder at a loss of 25 per cent. What did he pay for the sheep if his gain was £3, 17s. on the whole transaction?

27. A wine merchant mixes 2 casks of wine worth 12s. 6d. a gallon with one cask worth 18s. per gallon, and sells the mixture at 15s. per gallon. What profit per cent. does he make?

28. A grocer buys a chest of tea containing 180 lbs. at 3s. $7\frac{1}{4}$d. per lb. If 10 lbs. be spoiled, what does he gain per cent. by selling the remainder at 4s. 4d. per lb.?

29. A dairyman buys milk at $2\frac{1}{2}$d. per quart, dilutes it with water, and sells the mixture at 3d. per quart. His profits are 60 per cent. upon his outlay. How much water does he mix with each quart of milk?

30. A grocer buys $3\frac{1}{2}$ cwts. of tea at £13, 16s. per cwt., and 4 cwts. at £12, 12s. per cwt. He sells 7 cwt. at 2s. 9d. per lb. At what price per lb. must he sell the rest so as to gain $12\frac{1}{2}$ per cent. on the whole cost?

31. A person buys equal quantities of apples at the rate of 2 a penny and 3 a penny respectively, and then mixes them. How many may he then sell for 5s. so as neither to gain nor lose by the transaction?

32. If the price of candles $8\frac{1}{2}$ inches long be 9d. per half-dozen, and that of candles of the same thickness and quality, $10\frac{1}{4}$ inches long, be 11d. per half-dozen, which kind do you advise a person to buy?

33. A wine merchant buys two sorts of spirits at the rate of 16s. 6d. and 10s. 6d. per gallon. He mixes them in the proportion of 5 parts of the cheaper to 7 of the dearer. At what rate per gallon must he sell to make 25 per cent. profit?

34. If $5\frac{1}{2}$ per cent. would be gained by selling 121 yards of silk for £26, 11s. $10\frac{3}{4}$d., at what price per yard must it be sold to gain 12 per cent.?

35. A merchant buys 4000 quarters of corn, one-fifth of which he sells at a gain of 5 per cent., one-fourth at a gain of 10 per cent., one-half at a gain of 12 per cent., and the re-

mainder at a gain of 16 per cent. If he had sold the whole at a gain of 11 per cent. he would have made £72, 16s. more. What was the cost of corn per quarter?

36. A bankrupt has goods worth £975, and had they realized their full value his creditors would have received 16s. 3d. in the £; but $\frac{3}{8}$ were sold at 17.5 per cent., and the remainder at 23·75 per cent. below their value. What sum did the goods fetch, and what dividend was paid?

37. A man has been paying income tax at 3d. in the £. The tax is raised to 4d. in the £, but he is now allowed to deduct £180 from his income, paying tax on the remainder only. The total amount of tax he now pays is the same as before. Find his income.

38. A retail tradesman professes to charge 10 per cent. above the wholesale price, but he has adulterated his goods with 50 per cent. of an inferior kind, which only costs $\frac{2}{5}$ of the price. What is his real rate of profit?

39. After probate duty of £88 and lawyer's bill of £20, 6s. 8d. are deducted from a personal estate of £3616, 3s. 9d., the remainder is divided into five equal parts, one to a widow free of legacy duty, one to a sister who pays 1 per cent. duty, and one to each of the three nieces paying 3 per cent. If the above probate and legacy duties be commuted for a probate duty of $4\frac{1}{2}$ per cent., and the lawyer's bill be thereby reduced to £3, 15s., how much ought to have been left to each legatee so that the net sums received should be exactly the same proportion as before?

40. During the first half of the present financial year the income tax was fixed at 5d. in the £. But during the second half it was increased to 8d. in the £. A gentleman whose income was at the rate of £1000 a year more in the second than in the first half-year, finds that he pays for the year £35, 8s. 4d. more income tax than he would have paid if his income and the income tax had remained unchanged. Find income and income tax.

41. The income tax is reduced from $10\frac{1}{2}$d. to 5d., but a man's gross receipts are reduced by 10 per cent. Find by what percentage his net income is altered.

42. Find a constant multiplier which will change the

numbers of lbs. into the price expressed in sixpences, if 5 lbs. cost 3s.

43. What unit of weight in question 42 must be used so as to obtain an integral multiplier?

44. What unit of price must be used so as to obtain an integral multiplier?

45. What is the difference in price per dozen between selling eggs which cost me 1s. a score at 10 per cent. and 15 per cent. profit?

46. Give yourself the answer and other data, and find the buying price.

47. Find the better profit per cent.

48. Find the lesser profit per cent.

49. Find the number you buy for the shilling.

50. What did eggs cost me a hundred, if by reducing the number of eggs I sell for a shilling from 10 to 8 I increase my proft by $37\frac{1}{2}$ per cent.?

CHAPTER XIV.

Compounding Ratios—Compound Proportions worked out in different ways.

1. Ratios are compounded by multiplying their measures.

2. Sometimes a ratio depends on two or more other ratios. When this is the case, the measure of the single ratio is found to be equal to the measures of the others compounded, that is, multiplied together. Thus the ratio between the two prices of feeding two lots of sheep for two separate periods, depends both on the ratio of the number of sheep and of that of the number of weeks.

If 4 sheep, to feed 2 weeks, cost 4s., to feed 12 sheep for 5 weeks will cost 30s., since the ratio $\frac{4}{30} = \frac{2}{5} \times \frac{4}{12} = \frac{8}{60}$ or $\frac{4}{30}$.

3. The proof of this important proposition as shown in Algebra is by no means easy.

4. In arithmetic we may first assume that all the ratios are ratios of equality except the one we are considering, and state this particular ratio as in Simple Proportion. Supposing our question is—

If 24 cakes can be made out of 3s. worth of oatmeal when its price is 18d. per peck, how many cakes can be made out of 10s. $3\frac{1}{2}$d. worth of oatmeal when it is at 13d. per peck?

Here our incomplete ratio is one of numbers of cakes, which depends on the ratios of the prices of oatmeal, and the quantities of oatmeal made into cakes,—these latter quantities being expressed by their cost. Let us first assume that the quantities of oatmeal used are the same, and find the number of the cakes subject to the condition of the price of oatmeal. This is an inverse proportion, as the more valuable the oatmeal the less cakes could be made for the same money. Hence our first proportion is

13d. : 18d. = 24 cakes : number of cakes required.

Hence the result will be $24 \times \frac{18}{13}$ cakes.

Now our question is, If with 3s. worth of oatmeal we can make $24 \times \frac{18}{13}$ cakes, how many can we make with 10s. $3\frac{1}{2}$d.

worth? This is a direct proportion, and our statement is

$$3s. : 10s.\ 3\tfrac{1}{2}d. = 24 \times \tfrac{18}{13} \text{ cakes : number of cakes}$$

and our result is $24 \times \dfrac{18}{13} \times \dfrac{247}{3 \times 12 \times 2}$ cakes, or 114 cakes.

To state this altogether—

$$\left.\begin{array}{ll} \text{I.} & 13d. : 18d. \\ \text{D.} & 3s. : 10s.\ 3\tfrac{1}{2}d. \end{array}\right\} = 24 \text{ cakes : number of cakes to be found.}$$

The lines, over and under, connecting the corresponding terms, as shown in Chapter II.

5. Now to do this question by the unitary method.

If when oatmeal is 18d. a peck 3s. worth will make 24 cakes,

∴	„	1d.	„	3s.	„	24×18	„
and	„	1d.	„	1s.	„	$\dfrac{24 \times 18}{3}$	„
∴	„	13d.	„	1s.	„	$\dfrac{24 \times 18}{3 \times 13}$	„
and	„	13d.	„	$10\tfrac{7}{24}$s.	„	$\dfrac{24 \times 18 \times 247}{3 \times 13 \times 24}$	„

which gives us as before 114 cakes.

This method may easily be shortened by reducing the quantities which correspond to the known term of the incomplete ratio to unity at the same time, and also introducing together the terms corresponding to the unknown term of the incomplete ratio. Thus—

If when oatmeal is 18d. a peck 3s. worth will make 24 cakes,

then	„	1d.	„	1s.	„	$\dfrac{24 \times 18}{3}$	„
and ∴	„	13d.	„	$10\tfrac{7}{24}$s.	„	$\dfrac{24 \times 18 \times 247}{3 \times 13 \times 24}$	„

The Fractional Method.

Since we have to multiply the number of cakes by $\frac{18}{13}$ to find the number we could make, supposing we spent the same money, if the price of oatmeal changed from 18d. to 13d., and also by $\dfrac{\frac{247}{24}}{3}$ if the money we spent was changed from 3s. to $\frac{247}{24}$s.

Hence the number of cakes is immediately found by multiplying the number 24 by $\frac{18}{13}$ and by $\dfrac{\frac{247}{24}}{3}$, which gives us as before—

$$24 \times \frac{18}{13} \times \frac{247}{24 \times 3}, \text{ or } 114.$$

By the Method of Units.

To work this particular example by this method is a very useful mental exercise, but not a particularly practical one, being by no means easy.

To take such a unit that we may have no fractions in the working, we must take $\dfrac{24}{18 \times 13}$ of the quantity of oatmeal to be bought for a halfpenny.

Then we can buy 24×13 such units of the better oatmeal for $\frac{1}{2}$d., and 24×18 of the inferior. We therefore can get $24 \times 13 \times 72$ units of the better oatmeal for 3s., and this makes 24 cakes;

$\therefore$ there are 13×72 units in each cake.

For the other money we can get $24 \times 18 \times 247$ units; and since there are 13×72 units in each cake, we can obtain as many cakes as $24 \times 18 \times 247$ contains 13×72, or 114 as before.

As a mental exercise we strongly recommend the working out of many examples in this way. Practice will enable the student to determine on the best unit to use.

6. In Algebra we should first state that the number of cakes depended directly on the quantity of oatmeal used, and indirectly on the price of the oatmeal; and we should have to find some constant multiplier of the product of the quantity of oatmeal, and the reciprocal of the price of oatmeal. And the number of cakes will be found by multiplying 10s. $3\frac{1}{2}$d. $\times \frac{1}{18}$ by this constant multiplier, which multiplier is $\frac{24 \times 18}{3}$, or 144 when the unit of the price of oatmeal is a penny, and that of the value of oatmeal made into cakes shillings.

Thus the number of cakes which can be made of 15s. worth of oatmeal, which is worth 16d. a peck, is immediately found by multiplying $15 \times \frac{1}{16}$ by 144, which gives 135. This is the foundation of many unexplained rules often found in arithmetic.

7. Find a constant multiplier which will change any number of men into their wages when we know that 14 men earn £20 in 12 days. The amount of wages depends directly on the number of men and the number of days they work; hence the product of the number of men and the number of days must be multiplied by some multiplier, which in this case is $\frac{20}{14 \times 12}$, provided the unit of money is pounds, and that of time days. As an example: What are the wages of 10 men for 14 days?

Our answer is $\left(£10 \times 14 \times \frac{20}{14 \times 12}\right)$, which is £$16\frac{2}{3}$. By a change of unit we can make this constant multiplier almost what we like; for instance, by changing our unit of money from pounds to pence we reduce the denominator to 7.

Since it becomes $\frac{20 \times 20 \times 12}{14 \times 12}$, which is $\frac{200}{7}$, and by making it 200 sevenths of a penny we reduce it to unity.

This interesting but difficult subject will be more fully discussed when we treat of Proportion in connection with physical

problems, many of which only require a thorough knowledge of Proportion as taught in arithmetic to solve.

8. To detect an error in the placing of a ratio by the difference between the right answer and what we get. As an example: Supposing 4 men can plough 2 acres in 3 days, in how many days can 6 men plough 10 acres? The right statement is—

$$\begin{array}{ll} \text{I.} & 6 \text{ men} : 4 \text{ men} \\ \text{D.} & 2 \text{ acres} : 10 \text{ acres} \end{array} = 3 \text{ days} : \text{days to be found.}$$

$$\text{Ans. } \frac{4 \times 10 \times 3}{6 \times 2} \text{ days, or } 10 \text{ days.}$$

Now, supposing we had misplaced the men thus—

$$\begin{array}{l} 4 \text{ men} : 6 \text{ men} \\ 2 \text{ acres} : 10 \text{ acres} \end{array} = 3 \text{ days} : \text{days to be found,}$$

we should get

$$\frac{6 \times 10 \times 3}{4 \times 2}, \text{ or } 22\tfrac{1}{2}.$$

Now if we compare $22\frac{1}{2}$, our wrong answer, with 10, the right one, we find it $\frac{45}{20}$, or $\frac{9}{4}$, or $\frac{3 \times 3}{2 \times 2}$, which shows us that the 3 : 2 is wrongly placed. Since a 3 appearing in the numerator and disappearing in the denominator shows us that the fraction has been multiplied by 3 times 3, or 9; and for the 2 to disappear in the numerator and to appear in the denominator is dividing the fraction by 2 times 2, or 4; hence we know that the ratio 3 to 2, or 6 to 4, viz. that of the men, is wrongly placed. The ratio of the erroneous answer to the right one, provided, of course, that the error is alone caused by the misplacing the two terms of a ratio, will always be a square quantity; and if two ratios are misplaced, the product of two square quantities.

To take an example in which the incomplete ratio depends n several ratios.

If 3 men can dig a ditch 240 yds. long, 12 ft. wide, and ft. deep in 1000 days, working $8\frac{1}{2}$ hours a day, in how iany days of 10 hours could 17 men dig a ditch 20 per cent. ss long, half as wide again, and $\frac{1}{3}$ less deep, supposing that o of the latter men can do in the same time as much work as of the former? It is often worth while to write this down gain, thus—

men.	yds. long.	ft. wide.	ft. deep.	days.	hrs. a day.	power.
3	240	12	9	1000	$8\frac{1}{2}$	10
17	$\frac{4}{5}$ of 240	$\frac{3}{2}$ of 12	$\frac{2}{3}$ of 9	?	10	9,

f which the statement will be—

I. 17 men : 3 men.
D. 240 yds. : 192 yards = 1000 days : days to be found.
D. 12 feet : 18 feet.
D. 9 feet : 6 feet.
I. 10 hrs. : $8\frac{1}{2}$ hours.
I. 9 power : 10 power.

)f which the working is

$$\frac{3 \times 192 \times 18 \times 6 \times 17 \times 10 \times 1000}{17 \times 240 \times 12 \times 9 \times 10 \times \ 2 \times 9} \text{ days, or } 133\tfrac{1}{3} \text{ days.}$$

If asked how many hours the $\frac{1}{3}$ represents, of course the nswer is $3\frac{1}{3}$ hours, being $\frac{1}{3}$ of a 10 hour day. Now, by in-erting this term in the place where the question mark is put the arrangement adopted under the question, we can erase ny term and find it from the elements left. This one question, ien, could be asked in fourteen ways. If asked to find a con-ant multiplier to change one term into another, the term asked r must be one of the incomplete ratio, and the term which ultiplied must be put away from the fraction and all the rest ancelled. Thus, to find the constant multiplier, which will hange the number of yards in length into the number of men iat will be able to complete the work under the foregoing onditions, we must make the men the incomplete ratio, and nd the multiplier which will convert the 192 yards into 17

men (of course we are not supposed to know this 17), and if we have the 192—the number of yards to be multiplied—uncancelled, we find the multiplier will be $\frac{17}{192}$, which, of course, immediately gives us 17 men.

EXAMINATION AND EXAMPLES.

1. When is a proportion said to be compound, and why?

2. Split up the following question into two Simple Proportion questions. If 9 men will earn £8 in 20 days, in how many days will 6 men earn £10?

3. What do you understand by a constant multiplier to convert one element of a question into another?

4. If 4 men reap 5 ac. 3 ro. 39 po. in a week of 6 days, how many men at the same rate would reap 35 ac. 3 ro. and 34 po. in 12 days?

5. If 3 persons are boarded 4 weeks for £7, how many can be boarded 13 weeks 5 days for £112?

6. The cost of keeping 25 horses being at the rate of £11, 6s. 0½d. per week, what will be the cost of keeping 13 horses from September 1 to October 31?

7. If 7 bush. 2 pks. be consumed by 10 horses in 7 days, how many horses will consume 3 qrs. 6 bush. at the same rate in 10 days?

8. If a man walk 600 miles in 25 days, walking 8 hours a day, in how many days will he walk 330 miles, walking 10 hours a day?

9. If 5 horses eat 8 bush. 1¾ pks. of oats in 9 days, how long at the same rate will 66 bush. 3¾ pks. last 17 horses?

10. How many hours a day must 24 men work to accomplish as much in 5 days as 25 men could do in 4 days, if they worked 6 hours a day?

11. If 6 men can do a piece of work in 30 days of 9 hours each, how many men will it take to do 10 times the amount, if they work 25 days of 8 hours each?

12. If 8 horses plough $11\frac{2}{3}$ acres in 2 days, in how many *days* will 6 horses plough 17½ acres?

13. If, when meat is at 9d. per lb., it costs £11, 16s. 3d. to supply a family of 12 persons for 5 weeks, how much will it cost to supply a family of 18 persons for 7 weeks, when meat is at 1s. per lb.?

14. If the cost of printing a book of 320 leaves, with 21 lines on each page, and on an average 11 words in each line, be £19, find that of printing a book with 297 leaves, 28 lines on each page, and 10 words in each line.

15. How many hours a day must 171 men work to do in 91 days what it takes 143 men 133 days of 9 hours each to do?

16. If 8 men can build 2 roods of wall, one brick in thickness, in 3 days, how many men must be employed to build 5 roods, a brick and a half in thickness, in a week?

17. If the wages of 4 men for 12 days be £6, what would be the wages of 6 men for 10 days?

18. How many days of 12 hours each would 100 men take to perform a piece of work in, when 45 men can do the same in 16 days of 10 hours each?

19. If 5 men dig a trench in $1\frac{1}{3}$ days, working $4\frac{1}{2}$ hours a day, how long would one man take to dig a trench half as large again, working 5 hours a day?

20. If 15 men build a wall 25 yds. long, 2 yds. high, 2 ft. thick, in 14 days, in what time will 7 men build a wall 30 yds. long, 7 ft. high, and 1 ft. thick?

21. If 5 men can mow 33 acres in 5 days, working 11 hours a day, how many days would it take 4 men to reap 32 acres, working 10 hours a day, if they can reap 5 acres in the same time that they can mow 6?

22. If the wages of 54 men for 36 days amount to £60, 15s., what will be the wages of 30 men and 50 boys for 32 days, supposing 2 men to do as much work as 5 boys?

23. On a piece of work 3 men and 5 boys are employed who do half of it in 6 days. After this 1 more man and 1 more boy are put on, and $\frac{1}{3}$ more is done in 3 days. How many men must be put on that the whole may be completed in 1 day more?

24. If £3 are the wages of 4 men for $6\frac{1}{2}$ days, what are the wages of 17 men for 4 weeks of $5\frac{1}{3}$ days in each week?

25. If a certain MS. fill 6 printed sheets, each containing 32

pages, and each page 24 lines, with 9 words in each line, how many lines of 12 words each must there be in a page, in order that another similar MS. twice as long may fill 3 sheets, each containing 48 pages?

26. If 44 labourers can do a piece of work in 15 days of 10 hours each, how many navvies must be employed to do $\frac{3}{4}$ths more work in 7 days of 11 hours, supposing 3 navvies able to do the work of 5 labourers?

27. A copyist can transcribe 3 pages of a certain work in an hour and a half. How long will it take 3 men, working only half as rapidly, to copy 36 pages of another volume, the pages of which contain $\frac{1}{8}$ as much again as those of the former?

28. Supposing an arithmetic paper to be done by 2500 persons, of whom one-fifth are girls and the rest boys; and supposing 5 per cent. of the boys to fail and 40 per cent. of the girls to fail: find what percentage of the whole number of candidates succeed in passing.

29. If 4 men, working 7 hours a day, do a piece of work in 5 days and 1 hour, how many hours a day must 6 men work to do twice as much in 6 days and 4 hours, allowing for a half-holiday on one of the 6 days?

30. If it cost 16 guineas to supply 30 men, 48 women, and 60 boys with bread for a week, supposing a man to eat twice as much as a boy, and a woman to eat one-fourth less than a man, how much will it cost to supply 25 men, 60 women, and 72 boys for 10 days?

31. If 10 men and 15 boys can reap a field in 6 days, in how many days will 7 men and 12 boys reap it, supposing that 2 men can do as much as 3 boys?

32. Workmen can perform a certain labour in a week if they work 11 hours a day for 6 days. How many hours a day must they work to perform the same in the same time if they take half of Saturday as a holiday, but do a twelfth more work each hour?

33. Two houses are built; the number of workmen employed on them are as 4 : 3, but their wages are as 5 : 6, and the time they take 2 : 7. Find the ratio between the costs of the houses.

34. In 34 give yourself that the ratio of value of houses is o : 63. Find that of the times.

35. If there were 60 men employed on the first house, how ıany were employed on the latter?

36. Find the constant multiplier which will immediately onvert the number of men into the cost of the house they uild, expressed in £, the units being gangs of men of 25, and £10. Show that the other units need not be known.

37. Find the constant multiplier which will immediately onvert the number of months they work into their weekly ages expressed in shillings, the units being months and florins.

38. Two houses are being built by workmen whose numbers re as 3 : 2, their wages being 5s. and 4s. a day respectively; ut the former work 3 hours each day to the others' two, and . men of the latter can only do as much as 3 of the others. f the cost of the two houses is the same, find the ratio between heir times.

39. If the unit be a year of 310 days, and that of men gangs f 20, what did the houses cost?

40. A piece of work is finished in $45\frac{1}{3}$ hours by 3 sets of nen, numbering 5, 7, 2 respectively; 4 of the first lot can do s much as 5 of the second, and 3 of the third lot as much as of the second. Find the time in which 3 sets of men, numering 7, 5, 3 of the above powers, could finish it, the 7 men orking as well as the 5, the 5 as the 7, and the 3 as the 2.

41. How much a lb. must I sell tea which costs me £10 a wt., so as to gain 11 per cent. and the price of the carriage, per cent. on the cost price?

42. How many pounds' worth of tea did I buy, if by selling at a profit of $16\frac{1}{2}$ per cent. I make £21 more than if I old it at $15\frac{3}{4}$ per cent.?

43. How many roods of ditch can 100 men dig in 19 days f 8 hours each, if 20 men who are 25 per cent. better workıen can dig 100 roods in 20 per cent. less time?

44. If the second lot of workmen work 9 hours a day, how ıany days do they work?

45. If 120 men can build a house 60 ft. high in 15 days, ow many will it take to build a house one-twelfth less high, ıd in two-thirds of the time?

46. In a school of 1600 children, 56·25 are girls, who earn $\frac{9}{16}$ of the grant (£1440). If the number of failures of the boys be 7 per cent., find that of the girls.

47. If 5 per cent. be the result of the last question, give yourself it and the other data except the fraction of the grant that the girls earn, and find it.

48. Similarly, find the ratio of girls to boys.

49. Find the number of girls who pass, and the amount of grant to each child.

CHAPTER XV.

Compound Proportion, with Interest, worked out (1) by Proportion, and (2) by Decimals, and (3) in Ordinary Way, (4) Unitary Method.

1. If I ask for the interest on £200 for 3 years at 5 per cent. per annum, the real question is this: If I pay £5 for the use of £100 for 1 year, how much shall I pay for the use of £200 for 3 years? of which the statement is as follows, the incomplete ratio being connected *directly* with both the others:—

$$\begin{array}{ll} \text{D. } \pounds 100 : \pounds 200 \\ \text{D. } \ \ 1 \text{ yr.} : 3 \text{ yrs.} \end{array} = \pounds 5 : \text{interest required.}$$

The working of this is $\pounds\frac{200 \times 3 \times 5}{100 \times 1}$, or £30.

Now, I can find *any* one of these six terms provided I know the other five, though the £100 and the 1 year are not often asked for.

When asked for the 5 or rate per cent., the question is, At what rate per cent. per annum will the interest on £200 amount to £30 in 3 years?

The statement is as follows, in which the incomplete ratio is connected *directly* with the principals and *indirectly* with the time—

$$\begin{array}{ll} \text{D. } \pounds 200 : \pounds 100 \\ \text{I. } \ 3 \text{ yrs.} : 1 \text{ yr.} \end{array} = \pounds 30 : \text{rate required,}$$

giving, as working, $\pounds\frac{100 \times 1 \times 30}{200 \times 3}$, or £5.

Next, to find the principal. Here our question is, What sum of money will produce as interest £30 in 3 years at 5 per cent. per annum.

The statement is as follows, the incomplete ratio being connected *directly* with that of interest and *indirectly* with that of time:—

$$\begin{array}{ll} \text{D.} & £5 : 30 \\ \text{I.} & 3 \text{ yrs.} : 1 \text{ yr.} \end{array} = £100 : \text{principal required,}$$

of which the working is $£\frac{30 \times 1 \times 100}{5 \times 3}$, or £200.

Lastly, to find the time, when the question would be, In what time would £200 produce £30 as interest at the rate of 5 per cent. per annum? Here the incomplete ratio is connected *directly* with the interest and *indirectly* with the principals. Thus—

$$\begin{array}{ll} \text{D.} & £5 : £30 \\ \text{I.} & £200 : £100 \end{array} = 1 \text{ year} : \text{time required,}$$

and we get $\frac{30 \times 100 \times 1}{5 \times 200}$ years, or 3 years.

2. If we are asked for the interest for so many months or days, of course, as in any other question, the two terms of the ratio must be reduced to the same denomination.

3. Since the interest on £1 for a year at (say) 6 per cent. is $£\frac{1 \times 1 \times 6}{100 \times 1}$, or ·06, the interest on £2 for 1 year would be double this, and that on £50 for the same time 50 times this, etc., and the interest on £2 for 2 years would be 4 times this, and on £10 for 4 years 40 times this. We get a very simple way of finding the interest, viz. reduce the principal and time to pounds and years decimally, and multiply the product of these by the interest on £1 for 1 year at the given rate.

E.g. find interest on £50, 15s. for 2 years 3 months at 4 per cent.

£50, 15s. = £50·75,
2 yrs. 3 mo. = 2·25 yrs.,

and interest on £1 for 1 year at 4 per cent. = ·04.

$$\begin{array}{r} 50{\cdot}75 \\ 2{\cdot}25 \\ \hline 25375 \\ 10150 \\ 10150 \\ \hline 114{\cdot}1875 \\ {\cdot}04 \\ \hline £4{\cdot}567500 \\ 20 \\ \hline 11{\cdot}3500 \text{ shillings} \\ 12\phantom{\text{ shillings}} \\ \hline 4{\cdot}20 \text{ pence} \end{array}$$

Ans. £4, 11s. $4\frac{1}{5}$d.

It is better, as a rule, to multiply fractional quantities as vulgar fractions than as decimals. The working of the above would be—

$$50\tfrac{3}{4} \times 2\tfrac{1}{4} \times \tfrac{4}{100},$$

$$\text{or } \frac{203}{4} \times \frac{9}{\cancel{4}} \times \frac{\cancel{4}}{100} = \frac{1827}{400},$$

$$\begin{array}{r} £ \\ 4)\underline{18{\cdot}27} \\ 4{\cdot}5675, \end{array}$$

which, as before, gives us £4, 11s. $4\frac{1}{5}$d.

Since it is so easy to divide by a hundred, it is a mistake to cancel any *part* of it. Of course, if it will entirely cancel out it is better to do so, but not otherwise.

4. Interest can be easily worked by the unitary method.

Thus, to take the same example—

If £100 earns in 1 year £5;

$$\therefore \text{£}1 \quad \text{,,} \quad 1 \quad \text{,,} \quad \text{£}\frac{5}{100},$$

$$\text{£}1 \quad \text{,,} \quad 3 \quad \text{,,} \quad \text{£}\frac{5 \times 3}{100},$$

$$\text{and £}200 \quad \text{,,} \quad 3 \quad \text{,,} \quad \text{£}\frac{5 \times 3 \times 200}{100},$$

or £30.

As before, the last two steps can be taken together.

To find one of the other elements, if the interest is known.

For example, find the rate per cent.

£200 earns in 3 years £30,

$$\therefore \text{£}1 \quad \text{,,} \quad 1 \quad \text{,,} \quad \text{£}\frac{30}{200 \times 3};$$

$$\therefore \text{£}100 \quad \text{,,} \quad 1 \quad \text{,,} \quad \text{£}\frac{30 \times 100}{200 \times 3},$$

which gives us the £5 per cent.

5. The interest on £200 for 3 years at 5 per cent. being, as we found above, £30, £200 is said to *amount* to £230 in 3 years at 5 per cent. Sometimes in a carelessly-worded question this word 'amount' is not used technically, but the nature of the problem will generally show whether this is the case or not.

6. The decimal method is more useful in questions involving amount than in those only involving interest. £1 amounts to £1·05 at the end of a year at 5 per cent. per annum, and to £1·1 at the end of 2 years, and £1·15 at the end of 3 years, and so on. If, then, £1 amounts to £1·15 at the end of 3 years, £75 will amount to £75 × 1·15 at the end of 3 years. Working them in this way gives us the result direct, and also shows us how to find the principal when the amount and other elements are known. Then, since the amount is found by multiplying the principal by 1 + the time × the rate divided by 100, the principal will be immediately found from the amount by dividing it by this fraction (expressed decimally).

Thus: What principal will amount to £230 at 3 per cent. in 5 years? £1 is worth £1·15 at the end of the period; hence divide £230 by 1·15, and the quotient £200 will be the principal.

7. To find the other elements, both the amount and the principal being known. In what time will £375 amount to £431, 5s., interest being calculated at 3 per cent. per annum? We must in this case subtract the principal from the amount, and proceed as in paragraph 1.

8. Find the difference in interest on £3000 for 4 years at 5 per cent., and for 3 years at 7 per cent.

Let us find them out by fractional method.

$$£\frac{3000 \times 4 \times 5}{100 \times 1} = £600,$$

$$£\frac{3000 \times 3 \times 7}{100 \times 1} = £630\,;$$

hence the difference is £30.

Now let us give ourselves this £30, and find the principal. The question would be as follows:—

9. On what sum of money would the interest at 7 per cent. for 3 years exceed by £30 that calculated at 5 per cent. for 4 years?

Let us find the difference on £1. Thus the interest on £1 for 4 years at 5 per cent. is $£\frac{4 \times 5}{100}$, or $£\frac{1}{5}$.

The interest on £1 for 3 years at 7 per cent. is $£\frac{3 \times 7}{100}$, or $£\frac{21}{100}$,

$$\text{and } £\frac{21 - 20}{100} = £\frac{1}{100}\,;$$

∴ the difference on each £ is $£\frac{1}{100}$, and considering that there is a difference of £30 altogether, there must have been as many pounds as $£\frac{1}{100}$ is contained in £30, which gives us £3000.

Though in this case we should have saved fractions had we found out the difference on interest on £100, as a practice it is far better to work with the single pound.

10. Here is a far more difficult question. The interest on a certain sum of money at $5\frac{1}{2}$ per cent. for 4 years is the same as the sum of the interest on £180 of it at 3 years at 8 per cent., and that on the rest at $6\frac{1}{2}$ years at 3 per cent. Find the sum of money.

The interest on each £ when the money is all invested together is £$\frac{22}{100}$.

The interest on each £ of the £180 is £$\frac{24}{100}$;

$\therefore$ on this I receive £$180 \times \frac{2}{100}$ more than I should have had it been invested in the lump, or £$\frac{18}{5}$, but on the rest of the money I only receive for each £, £$\frac{19\frac{1}{2}}{100}$, or £$\frac{2\frac{1}{2}}{100}$ less than I should have had I invested this in the lump sum; hence I must invest as many pounds at this latter rate as will make up for the gain in the other case—that is, as many pounds as £$\frac{2\frac{1}{2}}{100}$ is contained in £$\frac{18}{5}$,

$$\text{or } \frac{18}{5} \times \frac{200}{5}, \text{ or } £144.$$

Hence the sum of money I invested is £180 + 144, or £324.

11. If we call the principal P, the amount A, the rate R, the time T, and the interest I, the following relations will be seen to have been proved in paragraph 1 :—

$$I = \frac{PRT}{100}, \; P = \frac{100I}{RT}, \; R = \frac{100I}{PT}, \text{ and } T = \frac{100I}{PR}, \; A = P + I,$$

$$\text{or } P + \frac{PRT}{100}, \text{or } \frac{100P + PRT}{100}, \text{or } \frac{P(100 + RT)}{100},$$

$$\text{and } P = \frac{A - 100}{100 + RT}.$$

12. Here is a problem which is far easier than it looks at

first sight. A sum of money will ***amount*** at a certain interest to £46 in $2\frac{1}{2}$ years, and to £48 in $3\frac{1}{3}$ years. Find the amount and the rate of interest. The difference in amount is the difference in interest for the two periods. Hence £2 is the interest of the principal for $\frac{5}{6}$ year $(3\frac{1}{3}-2\frac{1}{2})$. The interest then for $2\frac{1}{2}$ years is found by the following proportion :—

$$\tfrac{5}{6} \text{ years} : 2\tfrac{1}{2} \text{ years} = £2 : \text{interest for } 2\tfrac{1}{2} \text{ years},$$

which gives us $\dfrac{2\times 5\times 6}{2\times 5}$, or £6.

Hence the principal is £40 (£46 − 6). And to find the rate at which £40 will give as interest £6 in $2\frac{1}{2}$ years, we can either take the formula $R=\dfrac{100\ I}{PI}$, or reduce it to a question of Compound Proportion, as in par. 1, either of which will give us 6 per cent.

13. To find a constant multiplier which will immediately convert principal into interest at a given rate, say 4 per cent., for a given time, say 6 months.

Our statement is—

$$\left.\begin{array}{l}£100 : \text{principal}\\ 1 \text{ year} : 6 \text{ months}\end{array}\right\} = £4 : \text{interest required.}$$

$$\text{Interest} = \frac{6\times 4}{100\times 12}\text{principal.}$$

Constant multiplier is $\frac{1}{50}$, or ·02.

Hence the interest for every 6 months at 4 per cent. is the product of the principal and ·02, which gives the answer in pounds. To convert the principal in pounds into interest in shillings, we must multiply the number of pounds by $\frac{2}{5}$, and for every 3 months multiply by $\frac{1}{5}$.

EXAMINATION AND EXAMPLES.

1. Explain clearly what you understand by the expressions 8 per cent. per annum and $\frac{1}{2}$d. per 2s. a month.

2. State as a Compound Proportion question the following. Find the interest on £500 for 4 years at 4 per cent. per annum.

3. What is the difference between the interest on £500 for 2 years at 4 per cent., and for 3 years at 3 per cent.?

4. Given this difference and the rates and times, find the principal.

5. What is the difference on charging 5 per cent. per annum and 1d. in every £ per month?

6. Find the simple interest of £612, 10s. for 6 years at $3\frac{1}{2}$ per cent.

7. At what rate per cent. per annum will £612, 10s. produce £2, 0s. 10d. each month?

8. Find the simple interest on £127, 9s. $4\frac{1}{2}$d. at $3\frac{1}{8}$ per cent. per annum for 4 years 2 months.

9. In what time will £127, 9s. $4\frac{1}{2}$d. give £16, 11s. $11\frac{81}{128}$d. interest at $3\frac{1}{8}$ per cent.?

10. What sum of money will earn as interest £16, 11s. $11\frac{81}{128}$d. at $3\frac{1}{8}$ per cent. in $4\frac{1}{6}$ years?

11. A person puts out £1250 to interest at $4\frac{1}{4}$d. per cent. What annual income does he derive from it?

12. If a person derive an income of £53, 2s. 6d. from £2500, at what rate is it let out?

13. What is the interest on £250 for $2\frac{1}{2}$ years at $3\frac{1}{2}$ per cent. simple interest?

14. I derive an income of £250 from an investment which pays me 4 per cent. per annum. What is the amount I have lent?

15. Find the simple interest on £1585, 17s. 6d.. for $6\frac{2}{3}$ years at $4\frac{1}{2}$ per cent.

16. If from lending £2000 for 8 years I receive £420, at what rate is interest charged?

17. Find the amount of £3050, 10s. 6d. at 5 per cent. per annum for $3\frac{1}{4}$ years simple interest.

18. On what sum of money is £100 the difference between the interest calculated at 4 per cent. per annum and that at $3\frac{1}{2}$ per cent. for every 10 months?

19. What sum of money will amount to £350, 19s. $4\frac{1}{2}$d. in $6\frac{1}{2}$ years at $4\frac{1}{4}$ per cent. per annum simple interest?

20. Find the interest on 1d. for 100 years at 5 per cent. per annum.

21. In what time will £1 increase itself tenfold at 4 per cent. per annum?

22. At what rate per cent. per annum will a given sum of money double itself at simple interest in 30 years?

23. At what percentage shall I be enabled to double my money in 8 years?

24. Find the simple interest on £1505, 2s. $1\frac{1}{2}$d. for $5\frac{1}{2}$ years at 8 per cent.

25. Find the amount which must be put out at interest to produce £300 a year at 4 per cent.

26. In how many years would 1d. *amount* to £1 at 10 per cent.?

27. Find the difference between two sums of money, the one of which produces £100 each year at 4 per cent., and another which produces the same amount at 5 per cent.

28. If the difference between two principals is £500, which produce the same amount of interest, the one calculated at 4 per cent. and the other at 5 per cent., find the common interest.

29. What will 2s. 6d. amount to in 80 years at 4 per cent. simple interest?

30. Find a constant multiplier which will change the principal into interest when it is calculated at 5 per cent. per annum.

31. How can we change the unit of price so as to reduce this multiplier to unity?

32. Find a constant multiplier which will change the principal into interest, if the latter be calculated at 1d. on the shilling per annum.

33. In what time will the interest on £1000, if calculated at 5 per cent., be £10 greater than if it were calculated at 4 per cent.?

34. At what per cent. will the interest on £1200 for 4 years be greater by £20 than for 3 years 6 months?

35. Show that the interest on any sum of money, calculated at 1d. in the florin, will produce as many pence as that calculated on a sum 100 times as great at 10 per cent. will produce pounds.

36. At what rate per cent. would the interest for every month be as many pence as there are pounds in the principal?

37. At what rate per cent. would the interest for every six months be as many florins as there are pounds in the principal?

38. What rate per cent. per annum is $\frac{1}{4}$d. for every £ for every day?

39. To find the amount of any sum of money at $5\frac{1}{2}$ per cent. per annum for any number of years, multiply the principal by 1·055 as well as by the number of years. Explain this rule.

40. What principal at $4\frac{1}{2}$ per cent. for 10 years will amount to the same as if I had invested £200 of it at 6 per cent. for 7 years, and the rest at 8 per cent. for 6 years?

41. Pawnbrokers charge on small sums $\frac{1}{2}$d. for every 2s. or fraction of 2s. per month, and $\frac{1}{2}$d. for the ticket. What per cent. per annum does a man pay on 2s. 9d. for 6 months?

42. What is the amount of £1475 at $3\frac{1}{2}$ per cent. for $2\frac{1}{2}$ years, and I then let it out so as to make an income of £80, 4s. $0\frac{3}{4}$d.?

43. I have an income of £80, 4s. $0\frac{3}{4}$d., which I derive from money lent at 5 per cent. This money I have saved for $2\frac{1}{2}$ years at $3\frac{1}{2}$ per cent. What was the original sum?

44. I invest part of £1200 at 5 per cent., and the rest at 4, and thereby obtain as interest £57. Find the sums.

45. Two sums of money, one of them £5 more than the other; the greater is let at $3\frac{1}{2}$ per cent. per annum, and the other at $4\frac{1}{2}$ per cent.; £98, 15s. 6d. will repay them both at the end of the year. Find the sums.

46. What would repay them at the end of six months?

47. What principal will amount to £42998·1696 in 8 years at 20 per cent. simple interest? Express your answer in pounds decimally correct to 4 places.

48. In what time will £56, 5s. od. amount to £64, 2s. 6d. at $3\frac{1}{2}$ per cent. per annum simple interest?

49. What sum of money at $5\frac{61}{240}$ per cent. per annum simple interest will produce in 15 years the same amount of interest that £500 will produce in 3 years at 5 per cent. per annum?

50. In question 49, with its answer, ask yourself a question of which the answer would be $\frac{61}{240}$.

CHAPTER XVI.

Discount and Present Worth.

1. In the last chapter the term amount was defined as the principal added to the interest. Hence to find the amount we had to find the interest and add it on to the principal.

2. The terms principal and amount then are reciprocals of each other. If £105 is the amount of £100, £100 is the principal of the amount £105.

3. In this part of the subject we generally call the principal present worth, and treat the amount as the principal, which I think is very unfortunate and confusing.

4. The difference between the principal and the amount, which we call interest in the last chapter, being calculated on the principal, is in this chapter called discount, being calculated on the amount, which here, being the known quantity, is considered the principal.

5. If money is not due for some time, according to agreement, of course its present value is not worth as much as it will be by and by; its future value is really the *amount* of the present value, and the discount is the difference between the two values. Our problem, then, is practically to find the sum of money which will amount to another at some future time at some fixed rate of interest.

6. What is the present value of £53, due 6 months hence, at 12 per cent.? We may either work with £100 or with £1. The proportion to find the present worth is this. If £100 will amount to £106 (which, of course, must be found as in last chapter), what will amount to £53 in the same time at the same rate? Which is stated thus—

D. £106 : £53 = £100 : £ present value required,

which gives us £50.

7. Knowing the present value, we can immediately find the discount, as it is the difference between the two, but the discount can be found directly thus. Find the discount on £84 at 8 per cent. for 4 months.

£100 will amount to £$102\frac{2}{3}$ in 4 months if the interest is calculated at 8 per cent.; therefore the discount on £$102\frac{2}{3}$ is £$2\frac{2}{3}$, and our proportion becomes

D. £$102\frac{2}{3}$: £84 = £$2\frac{2}{3}$: discount required,

which gives us £$\frac{3 \times 84 \times 8}{308 \times 3}$, or £2, 3s. $7\frac{7}{11}$d.

8. The student ought to notice that the interest on the present value is the same as the discount on the future value.

9. When bankers discount bills they charge interest on the future value, which is larger than the interest on the present value.

10. Considering that the, what we technically call interest, is interest on the present value, and discount that on the future value, the difference between the interest and the discount is the difference of interest on the present value and the future value, but the difference between the present and future values is the discount. Hence the difference of the interest and the discount is the interest on the discount, and the interest is the *amount* of the discount. The student is earnestly recommended thoroughly to grasp these ideas, as difficulties of this subject mostly arise from a misconception of the terms and their meanings.

11. To work another example of discount. Find the discount on £350 discounted 3 months before due at 4 per cent. £100 is worth £101, 3 months hence at 4 per cent.; ∴ the discount on £101 is £1. Hence our proportion is—

£101 : £350 = £1 : discount required,

which gives us £3, 9s. $3\frac{69}{101}$d.

The banker's discount or interest is found by this proportion.

£100 : £350 = £1 : interest required,

which gives us £3, 10s.

Hence the banker's gain is ·$8\frac{32}{101}$d., and this ought to be the interest on the discount. Let us see whether it is. This is our proportion—

£100 : £1 = £3, 9s. $3\frac{69}{101}$d. : interest required,

which will be found to give as before $8\frac{32}{101}$d.

12. To find the present worth by the unitary method.

Since £1 for 3 months at 4 per cent. amounts to £1·01,

ie amount of money that will amount to £350 at the same ite for the same time is

$$£350 \div 1\cdot01, \text{ or } £346,\ 10s.\ 8\tfrac{32}{101}d.;$$

id therefore our discount is £3, 9s. $3\frac{69}{101}$d.

13. To find it by the fractional method is far more racticable.

Since £100 is worth £101 in the given time at the given ite, the present worth will evidently be found by multiplying ie sum by $\frac{100}{101}$. And since the discount on £101 is £1 in ie given time at the given rate, the discount will be found by iultiplying the sum of money by $\frac{1}{101}$.

14. If the discount calculated at one rate per cent. is known, iat calculated at another rate for the same time (which must, owever, be known) can be found from it.

The discount on any sum of money, say £250, for 4 months t 5 per cent. is $£250 \times \dfrac{\frac{1}{3} \times 5}{100 + \frac{1}{3} \times 5}$, and as the same amount of ioney for 4 months at 6 per cent. is $£250 \times \dfrac{\frac{1}{3} \times 56}{100 + \frac{1}{3} \times 6}$, and iese sums of money are in the same ratio as

$$\frac{5}{100 + \frac{1}{3} \times 5} : \frac{6}{100 + \frac{1}{3} \times 6};$$

ence, if I know the value of either of the former quantities, can get the other from it, provided I know the time, but ithout necessarily knowing the principal.

Thus, if I know that the discount on a given sum of money or 5 months is £10 at 4 per cent., what will it be for same me at 6 per cent.?

The working will be this—

$$\frac{4}{100 + \frac{5}{12} \times 4} : \frac{6}{100 + \frac{5}{12} \times 6} = £10 : \text{discount required.}$$

$$\frac{12}{305} : \frac{12}{205} = £10 : \text{discount required.}$$

$$£\frac{12 \times 305 \times 10}{205 \times 12} = £14,\ 17s.\ 6\tfrac{30}{41}d.$$

15. If the difference between the discount and the interest be known, and the rate and time, the principal can be found.

Since, as we have proved, the difference between the discount and the interest is the interest on the discount, the problem is reduced to,—Given the interest, the rate, and the time, find the principal.

To work out an example. Given that the bill has 3 months to run, and is discounted at 4 per cent., and that the difference between the interest and the discount is $1\frac{91}{101}$d. Find the principal.

Since $1\frac{91}{101}$d. is the interest on the discount,

$$\therefore \text{ the discount is } \frac{100 \times 1\frac{91}{101}}{\frac{1}{4} \times 4}\text{d.} = \frac{19200}{101}\text{d.},$$

and the interest is this $+\frac{192}{101}$d.,

$$\text{viz. } \frac{19392}{101}\text{d.};$$

and this is the interest on this principal,

$$\text{viz. } \frac{100 \times \frac{19392}{101}}{4 \times \frac{1}{4}} = \frac{1939200}{101}\text{d.};$$

and $\frac{1939200}{101}$d. is £$\frac{1939200}{101 \times 12 \times 20}$, or £80.

Examination and Examples.

1. State clearly the difference between interest and discount.

2. Find the present worth of £108 due 2 years hence at 4 per cent. per annum.

3. How much greater is the interest, or, as it is sometimes called, the banker's discount, than the discount?

4. Find the discount on £252, 10s. due 4 months hence at 3 per cent.

5. On what sum of money is 15s. $10\frac{10}{101}$d. the discount when discounted at 4 per cent. 3 months before the amount is due?

6. If the discount on a sum of money for 4 months be £5 when discounted at 6 per cent. per annum, what will it be at 5 per cent. for same time?

7. A banker charges 1s. 6d. less than he might for discounting a bill at 4 per cent. which had 3 months to run, what was its amount?

8. The difference between the interest and the discount is £1, the rate per cent. 3, and the time 6 months, find the principal.

9. Show clearly that the difference between the interest and the discount is the interest on the discount.

10. If the present worth be $\frac{97}{100}$ of the future value, show that the future value is $\frac{100}{97}$ of the present worth.

11. If the present worth of a note due 4 months hence of £101 be £100, at what percentage is it discounted?

12. If the discount at a certain rate and for a certain time be found by multiplying the sum to be discounted by $\frac{1}{21}$, what fraction of the sum to be discounted is the interest calculated for the same time and at the same rate?

13. If the interest at a certain rate for a certain time be found by multiplying the principal by $\frac{1}{10}$, find the multiplier which will give the discount.

14. If the interest on £1000 for a given time and a given rate be £20, find the discount for the same time and at the same rate.

15. The difference between the interest and the discount is $\frac{1}{420}$ of the sum of money discounted, and the rate is 4 per cent. Find the time.

16. The interest on a sum of money for 6 months at a certain rate is £1, 5s. $2\frac{2}{5}$d., the discount on the same sum for same time is £1, 4s. $9\frac{27}{29}$d. Find principal and rate per cent.

17. On what sum of money is the discount 2s. less than the interest if they both be calculated for 4 months at 4 per cent.?

18. Show that the future value is the *amount* of the present worth.

19. The difference between interest on a certain sum for 6 months at 3 and 4 per cent. is £1, 6s. Find the sum.

20. The difference between the discount on a certain sum for 6 months at 4 and 6 per cent. is £$2\frac{1498}{1751}$. Find the sum.

21. What is the present worth of £276, 10s. 5d. due 219 days hence at $3\frac{1}{2}$ per cent.?

22. What is the amount of £270, 16s. 8d. at $3\frac{1}{2}$ per cent. for 219 days.?

23. Find the discount on £328, 13s. 5d. due 3 months hence at 4 per cent. per annum.

24. If the discount on £328, 13s. 5d. due 3 months hence be £3, 5s. 1d., at what rate is the discount calculated?

25. Find the interest on £328, 13s. 5d. for 3 months at 4 per cent., and show that the difference between it and £3, 5s. 1d. is the interest on £3, 5s. 1d. for the same time at the same rate.

26. Show that at 5 per cent. the interest on £650 for 3 months is equal to the discount on £495, 12s. 6d. due in 4 months' time.

27. There is a sum of money on which the interest at 4 per cent. for 3 months is 12s. more than the discount for same time. Find it.

28. Find discount on £929, 10s. due $2\frac{1}{4}$ years hence at $2\frac{1}{2}$ per cent.

29. How long has a bill to run if the discount on £929, 10s., calculated at $2\frac{1}{2}$ per cent., is £49, 10s.?

30. If the discount on a bill, discounted at $4\frac{1}{4}$ per cent. per annum $4\frac{1}{2}$ months before it is due, be £11, 13s. 9d., what is the amount of the bill?

31. What is the simplest way of proving a discount question?

32. What sum of money paid down will discharge a debt of £1000 due in two equal half-yearly instalments, interest being reckoned at 5 per cent. per annum?

33. When would the £1000 be due in one sum if the discount at the same rate were the same?

34. Show that the interest on £450 for 7 months at 4 per cent. per annum is equal to the discount on £460, 10s. 0d. for the same time at the same rate per cent.

35. Explain why these two sums are equal.

36. What sum of money will amount to £83, 17s. $1\frac{17}{40}$d. in 10 years at $3\frac{1}{4}$ per cent.?

37. Find the difference between the simple interest and the discount on £100 for 5 years at 5 per cent.

38. If the present worth of £218 due 2 years hence is £200, what is the present worth of £1000 due 6 years hence at the same rate?

39. Find the discount on £793, 14s. $10\frac{1}{2}$d. due 9 months hence at 4 per cent.

40. Prove the last question by showing that the present value is the amount of £793, 14s. $10\frac{1}{2}$d.

41. Find the discount on £237, 10s. paid 2 years before it becomes due at 7 per cent. simple interest.

42. Find the discount on a bill of £10 due at the end of a year at 10 per cent.

43. If the discount on a certain sum of money for a given time, at a certain rate, is to the interest on the same sum for the same time at the same rate as 50 : 53, prove that the rate × by the time is = 6. Hence, if you know one, you can find the other.

44. Find a constant multiplier which will change the sum into its discount when the interest is calculated at 4 per cent. and the bill has 15 months to run.

45. Find a constant multiplier which will change the future value into the present if the bill have 3 months to run and interest is calculated at 6 per cent.

46. Find a constant multiplier which will change a sum of money into its amount 2 years hence at 3 per cent.

47. Find from the last question the present value of £2120 due 2 years hence at 3 per cent.

48. The interest on a certain sum of money calculated at $4\frac{1}{2}$ per cent. for 7 months is 18s. $7\frac{797}{900}$d. Find the discount on the same sum for the same time at the same rate.

49. Find the discount for the same time and at the same rate on the amount of £124, 13s. 4d. for $7\frac{1}{2}$ years at 5 per cent.

50. Find a constant multiplier which will change the amount of a bill discounted for 1 month at 3 per cent. into its discount. Will the multiplier be double for 2 months?

CHAPTER XVII.

Some Properties of Numbers in Proportions—The Lever, and Proportional Parts.

1. We have already shown how to reduce a fraction to another whose terms differ by a given number.

2. If two fractions or ratios are of the same value, we can introduce any multipliers, and add or subtract the products to form new numerators or denominators separately or together, provided we do exactly the same to both fractions or ratios, and the resulting fractions will still remain equal.

Thus, $\frac{3}{5}=\frac{6}{10}$,

$$\frac{3\times9-5\times4}{3\times8+5\times2}=\frac{6\times9-10\times4}{6\times8+10\times2},$$

which the student can verify for himself.

It will be noticed that to form the numerators we multiplied the numerators by 9 and the denominators by 4, and subtracted them, and that for the denominators we multiplied the numerators by 8 and the denominators by 2, and added them.

The proof of this particular example is this—

Since $\frac{3}{5}=\frac{6}{10}$,

$$\therefore\ \frac{3}{5}\times\frac{9}{4}=\frac{6}{10}\times\frac{9}{4}.$$

From each of these equals subtract 1.

$$\frac{3\times9}{5\times4}-1=\frac{6\times9}{10\times4}-1,$$

$$\text{or }\frac{3\times9-5\times4}{5\times4}=\frac{6\times9-10\times4}{10\times4}.$$

$$\text{Similarly, }\frac{3\times8}{5\times2}=\frac{6\times8}{10\times2}.$$

By adding 1 to each fraction, and reducing, we get

$$\frac{3\times 8+5\times 2}{5\times 2}=\frac{6\times 8+10\times 2}{10\times 2}.$$

Dividing these equals, the one by the one and the other by the other, the quotient must be equal;

$$\therefore \frac{(3\times 9-5\times 4)5\times 2}{(3\times 8+5\times 2)5\times 4}=\frac{(6\times 9-10\times 4)10\times 2}{(6\times 8+10\times 2)10\times 4},$$

$$\text{but } \frac{5\times 2}{5\times 4}=\frac{10\times 2}{10\times 4};$$

$\therefore$ dividing by these equals, we get

$$\frac{3\times 9-5\times 4}{3\times 8+5\times 2}=\frac{6\times 9-10\times 4}{6\times 8+10+2}.$$

This gives us a way of dividing a number into two parts, so that some multiple of the one part, together with some multiple of the other, is equal to some given number.

E.g. to divide $13\frac{1}{3}$ into two parts, so that 4 times one part and 6 times the other are together equal to 65. We have to find a fraction, which—*to explain the reason, not to work the sum*—we will call $\frac{n}{d}$. Of it we know that $\frac{n+d}{4n+6d}=\frac{13\frac{1}{3}}{65}$; hence, by what is shown above,

$$\frac{n+d}{4n+6d-4(n+d)}=\frac{13\frac{1}{3}}{65-53\frac{1}{3}}=\frac{13\frac{1}{3}}{11\frac{2}{3}},$$

$$\text{or } \frac{n+d}{2d}=\frac{13\frac{1}{3}}{11\frac{2}{3}} \text{ or } d=5\tfrac{5}{6},$$

$$\text{and } n=7\tfrac{1}{2}.$$

Now, if we reduce this to a rule, it simply comes to this: Multiply the original number by the smaller of the two multipliers, and subtract the product from the given sum of the multiples of the parts, and divide this difference by the difference between the two multipliers. Thus, divide $15\frac{1}{2}$ into two

parts, so that 6 times the one and 12 times the other are equal to 160. To apply the rule—

$$160 - 15\tfrac{1}{2} \times 6 = 160 - 93, \text{ or } 67,$$
$$67 \div 6 = 11\tfrac{1}{6}, \text{ one of the parts,}$$
$$\text{and } 15\tfrac{1}{2} - 11\tfrac{1}{6} = 4\tfrac{1}{3}, \text{ the other part.}$$

The common-sense reason of this is plain. If from 160 I take away $15\frac{1}{2} \times 6$, I take away one part $\times$ 6 and the other part $\times$ 6; therefore there is the part, to be multiplied by 12, taken 6 times, left.

This enables us to solve such questions as: How many lbs. of tea, at 3s. 6d. and 4s. 3d. respectively, must be mixed together so as to sell the mixture for £100 at 4s. a lb.?

I have to divide 500 (the number of lbs. at 4s. worth £100) into two parts, so that $3\frac{1}{2}$ times the one part and $4\frac{1}{4}$ times the second is equal to 100×20, or 2000.

Here $\dfrac{2000 - (3\frac{1}{2} \times 500)}{\frac{3}{4}\text{ (diff. between 4s. 3d. and 3s. 6d.)}}$ = number of lbs. at 4s. 3d., which will give us $333\frac{1}{3}$ lbs. at 4s. 3d.; and hence there are $166\frac{2}{3}$ lbs. at 3s. 6d., which the student can verify.

This also shows us how to divide any number, as 16, into two parts which are in any ratio, as 3 : 5, or of which one is $\frac{3}{5}$ of the other.

3. Of course the ordinary method of adding the numbers together for the denominator, and taking each number as the numerator of a fraction, and then finding this fraction of the number to be divided, can be explained in a common-sense way; but it really depends on the theorem above.

The common-sense explanation is something as follows: When a quantity is to be divided between A and B in the ratio of 3 to 5. If I take 8 of the units, I must give 3 to A and 5 to B; hence, as many eights as there are in the number so many threes must A have, and so many fives B; but to find the number of eights we must divide by 8, and this quotient will tell us how many threes A will have and fives B will have, or, in other words, A has $\frac{3}{8}$ of the quantity, and B $\frac{5}{8}$.

Let the number be 64; A will have 24, and B 40.

The working by the other method, though useful as a mental *exercise, is not* so easy. To divide 64 into two parts in

the ratio of 3 : 5. This is to find a fraction $= \frac{3}{5}$ whose terms added together are 64.

Let 64 be the denominator of the fraction obtained by adding together the terms of the fraction required, then the equivalent fraction to this one whose denominator is 64 must have as its denominator $3 + 5$, and for the numerator we can take which we like—the 3 or the 5, remembering that if we take the numerator of $\frac{3}{5}$, we find the numerator of the required fraction, and if the denominator, the denominator.

Let us take the numerator.

$$\text{Then } \frac{3}{3+5} = \frac{\text{some numerator}}{64},$$

$$\frac{3}{8} = \frac{\frac{3}{8}}{1} = \frac{\frac{3 \times 64}{8}}{64} = \frac{24}{64};$$

$$\therefore \frac{3}{8-3}, \text{ or } \frac{3}{5} = \frac{24}{64-24} = \frac{24}{40};$$

hence, as before, the numbers are 24, 40.

4. Problems on levers and moments are simply questions in Proportion, and may be introduced here. Of course the proofs of the mechanical theorem will not be introduced, as being altogether foreign to the work. They must be here treated as axioms.

Def.—A lever is a rigid rod supported at some point in it, called its fulcrum, with a force acting at one point of it, and the work to be done applied at another.

A poker is a simple example of a lever. Where the poker touches the bars of the grate is its fulcrum; where the hand grasps it is where the power acts; and where it touches the coals, the point where the work is done.

The distances from the fulcrum to where the power and the work are situated, are called the arms of the lever.

Here is the mechanical fact which, as I said, must here be taken as an axiom. The power and the work done are connected inversely with the lengths of the arms to which they are respectively applied. Let us call the power P, the work W.

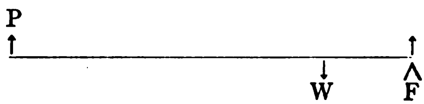
Then the following proportion is true:—

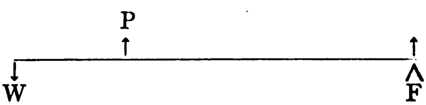
$$\text{I. } P : W = \text{arm to } W : \text{arm to } P.$$

Hence, if we know any three of these we can find the other; or if we know the sum of P and W, we can find them.

5. The following diagrams will show that there can be three kinds of levers, the arrows showing the direction in which the forces act:—

First kind


Second kind


Third kind


Familiar examples of these are—

First kind—crowbar, scissors.

Second kind—oar, bellows.

Third kind—arm, sugar-tongs.

In the first kind the pressure on the fulcrum is $P + W$.

" second " " " $W - P$.

" third " " " $P - W$.

In the first kind the power may be greater or less than the weight; in the second kind the power must be less than the weight; and in the third kind the power must be greater than the weight.

Let us work an example.

The pressure on the fulcrum of a lever of the second kind is 10 lbs.; the lengths of the arms are 2 feet and 3 feet. Find the power and the weight.

Since the pressure on the fulcrum is $W - P$, which is equal to 10, we have to find a fraction $= \frac{2}{3}$ whose terms differ by 10.

Hence we have to find a fraction whose denominator is $10 = \dfrac{2}{3-1}$.

$$\frac{2}{1} = \frac{20}{10};$$
$$\therefore \frac{2}{3} = \frac{20}{30}.$$

Hence, supposing the power to be the less, it is 20 lbs. applied 3 feet from the fulcrum, and the weight 30 lbs. applied 2 feet from the fulcrum.

If we state this as a proportion, we have

$$3 \text{ ft. (arm of power)} : 2 \text{ ft. (arm of weight)}$$
$$= 30 \text{ lbs. (W)} : 20 \text{ lbs. (P)};$$

and if we remember the property of a proportion, that the produce of the extremes is equal to that of the means, we get

$$3 \times 20 = 2 \times 30,$$

or the P × its arm = W × its arm.

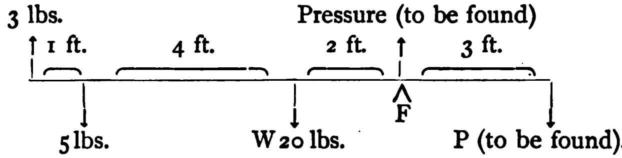
Of course, from one at least of the factors of each of these products we must abstract the denomination, and treat them as numbers, to enable us to read the result; but it is better to abstract both the denominations.

These products, viz. the weight expressed by a unit (here a lb.) × a distance expressed by a unit (here a foot), are called the moments of the force about the point (here the fulcrum) from which the points of application are measured.

These two moments act in opposite directions, and, being equal, no motion takes place. If you refer to the figure of the second kind of lever, you will see that the P tends to pull the lever round F, the way that the hands of the clock go, and might be termed clockwise, whereas the W tends to pull the lever round the other way, or non-clockwise.

It must here be taken as an axiom, that if any number of forces act on a rigid body so as to produce no motion, the sum of all the moments of these forces round any point acting clockwise, is equal to the sum of the moments of the forces acting non-clockwise.

We will take one example, and draw a figure. *Note.*—The weight of a body acts through its centre of gravity, that is, where it would balance, which, if not mentioned otherwise, is supposed to be at the middle point.



There is a bar 10 ft. long, and weight 20 lbs., supported at a point 3 ft. from one end. 9 ft. from this end there is a weight of 5 lbs. attached to the bar. At the end of the bar, 7 ft. from the fulcrum, there is an upward force of 3 lbs. exerted. What force P must be exerted at the other end so that the bar may remain at rest, and what pressure does the fulcrum sustain?

Taking moments round the fulcrum, those acting non-clockwise are—

W 20 lbs. acting 2 ft. from F, or a moment of 40,
and 5 lbs. „ 6 ft. „ „ 30.

These are partially counterbalanced by 3 lbs. acting clockwise 7 ft. from F, giving a moment of 21; hence the moment of P round F must be $70-21$, or 49; hence P will be found by dividing 49 by 3, or $16\frac{1}{3}$ lbs., and the pressure on the fulcrum is $5+20+16\frac{1}{3}-3$, since the three former act downwards and the 3 lbs. acts upwards, and, of course, the upward pressure is exactly equal to the downwards. Now, knowing the upward pressure exerted by the fulcrum, viz. $38\frac{1}{3}$ (which, of course, we could replace by a force of that amount acting at that point), and the amount of the force P, we can find any one of the other elements, and take moments round any point we like. *E.g.* let us find the distance from the fulcrum that the 5 lbs. acts, and let us take moments round the point where the 3 lbs. acts. The moment of 3 lbs. round where it acts is, of course, zero, since 3×0 (the distance) $= 0$. Students often think that on this account, that had we had a 5 lbs. or any other weight here, the result would be the same, but the 3 has affected the pressure on F. Taking moments round the point where the 3 lbs. acts, we have clockwise $5 \times$ unknown distance $+ 20 \times 5 + 16\frac{1}{3} \times 10$, or $263\frac{1}{3}$ and $5 \times$ the unknown distance, and the only force acting non-clockwise is the upwards pressure of the fulcrum, viz. $38\frac{1}{3}$,

acting 7 ft. from the point where the 3 lbs. acts, or $258\frac{1}{3}$; hence the moment of the 5 lbs. is 5, or it acts at 1 foot from where the 3 lbs. acts.

The student can find any of the nine elements in this problem, and vary the problem at will by taking moments at any point he likes in the line, even if produced.

If we know all the weights and pressures but one, we can, of course, find the remaining one by simple subtraction, remembering that the downward pressures must equal the upward ones. In working the example, the student is recommended always to draw a figure and insert the arrows, and consider the point round which moments are being taken as the centre of a clock face.

Examination and Examples.

1. Find the fraction $= \dfrac{3}{3+5}$ whose denominator is 24.

2. Divide 48 lbs. amongst two persons in the ratio of 3 : 5.

3. Divide 100 into two parts in the ratio of $\frac{1}{2}$ to $\frac{1}{3}$.

4. Could question 3 be asked in a different way?

5. Divide 203 into two parts in the ratio of $\frac{2}{3}$: $\frac{5}{7}$.

6. How much tea at 4s. a lb. must be mixed with tea at 5s. a lb. so as to form a mixture of 180 lbs. in which the value of the different teas are equal?

7. Divide £105 between two persons, so that the one has 1s. 1d. as many times as the other has 1s. 3d.

8. Divide $15\frac{1}{2}$ into two parts, so that 8 times the one part and 12 times the other may be equal to 149.

9. Divide £10 into two sums, so that 11 times the one and 13 times the other will equal £120.

10. Divide 10 into two parts, so that *the difference* between 5 times the one part and twice the other is equal to 8. There will be two pairs of numbers which will satisfy the conditions.

11. Give a common-sense explanation of the method of solving question 10.

12. Divide 25 into two parts, so that the difference between 3 times the one part and 4 times the other is 2.

13. Find two numbers whose difference is 2, so that the product of the lesser number and 5 less the product of the other and 4 is equal to 102.

14. What limit is there to the relative amounts of the different elements in the above question to render it possible?

15. Is it possible to find two numbers whose difference is 7, so that 5 times the greater is greater than 3 times the less by 4?

16. What is the least difference between the multiples possible in question 15?

17. Divide 16 into two parts, so that 5 times the one and 8 times the other are equal to 100.

18. What limits are there to the relative value of the elements of these problems?

19. Are there two answers to problems of this kind?

20. Divide 10 into two parts, so that the sum of the one part multiplied by 3, and the other part multiplied by 4, is greater by 2 than the difference of the greater part multiplied by 3 and the other part by 4.

21. What numbers of lbs. of tea at 2s. 6d. and 2s. 9d. must be mixed that they will produce a mixture worth 2s. 8d.?

22. A chest of tea of 150 lbs. is sold partly at 2s. and partly at 2s. 2d. a lb., so as to produce £15, 15s. What amount of tea is sold at 2s. per lb.?

23. How many gallons of brandies, worth 16s. 6d. a gallon and 18s. 6d. a gallon, must be mixed so as to sell at 17s. a gallon?

24. How much adulterated brandy, in which the water is to the brandy as 1 : 15, must be mixed with a gallon of pure brandy so that the water may be to the brandy as 1 : 31?

25. If a tea merchant adulterates his coffee, which could be sold pure for 2s. 1d. per lb. at a profit of 15 per cent., with 10 per cent. of chicory, worth 1d. a lb., and sells it at 1s. 10d., what is his gain per cent.?

26. What is the length of the lever if the shorter arm is 3 ft., and the W : P as 3 : 5?

27. What weight at a distance of 10 ft. from the fulcrum *will exactly* balance a weight of 100 lbs. 2 ft. from the fulcrum?

28. The pressure on the fulcrum is 20 lbs., and the arms are as 5 : 7. Find the power, supposing it to be less than the weight.

29. What power applied 100 ft. from the fulcrum will exactly balance a weight of 1000 lbs. 3 ft. from the fulcrum?

30. A beam 32 ft. long and weight 100 lbs., whose centre of gravity is 10 ft. away from the thicker end, is carried by two men, the beam resting 1 ft. away from the two ends on the men's shoulders. What weight does each carry?

31. A beam AB, 8 ft. long, weight 100 lbs., rests on a fulcrum 3 ft. from B, has 1 lb. weight hung from a place 4 ft. from B, 2 lbs. hung 5 ft. from B, 4 lbs. hung 6 ft. from B, and 8 lbs. at A. Find the force, 1 ft. from B, which will support them all.

32. What is the pressure on the fulcrum?

33. Supposing the weight at A to be removed, and replaced by a force acting in the opposite direction; find the force at B which will keep the forces at rest.

34. Given the pressure on the fulcrum as 17 lbs., and an upward pressure at A (in a straight lever AB, 6 ft. weight 2 lbs.) to be 5 lbs., how can I place 22 lbs. so as to make the lever balance 2 ft. from A?

35. Find the fulcrum, if weights of 4 lbs. and 10 lbs. are hung at the two ends of a lever 6 ft. long of 10 lbs. weight.

36. A lever AB of 5 ft. will exactly balance 2 ft. from A; if placed 3 ft. from A, it will require 10 lbs. at B to make it balance. What is its weight?

37. Find the length of a lever AB, weight 8 lbs., if a force of 1 lb. at A will exactly balance 100 lbs., the fulcrum being 4 ft. from B.

38. What kind of a lever is a pair of tongs?

39. Divide 10 into two parts, so that 4 times the one is equal to 5 times the other.

40. Divide 17 into two parts, so that 6 times the greater part is 2 less than 7 times the smaller part.

41. Divide 17 into two parts, so that 6 times the greater part is 2 more than 7 times the smaller part.

42. Divide 17 into two parts, so that 7 times the greater part is 2 greater than 6 times the less.

43. In what ratio must I mix diluted brandy of which $\frac{1}{25}$ is water, with that of which $\frac{1}{30}$ is water, so as to make a mixture of which $\frac{1}{28}$ is water?

44. Divide amongst A and B £317, 4s., so that A has as many shillings as B pence.

45. Divide £250 amongst 100 children, of which there are $\frac{3}{5}$ boys, so that what is divided amongst the boys is to what is divided amongst the girls as 2 : 3.

46. Compare what each boy receives with what each girl receives.

47. A heavy beam AB, 18 ft. long, is resting on a fulcrum 7 ft. from A. If a man who weighs 112 lbs., sitting 1 ft. from A, makes the beam exactly balance, how heavy is the beam?

48. The arms of a pair of scales are as 99 to 100. What will a true lb. seem to weigh in either pan?

49. How many feet from A in AB, a weightless lever of 10 ft., must the fulcrum be placed so that 10 lbs. at A will balance 2 lbs. at B?

50. ABCDEFG is a lever weighing 100 lbs. AB = 1 ft., BC = 2, CD = 3, DE = 4, EF = 5, FG = 6; weights of a lb. are hung at all the points mentioned above. Where must I place the fulcrum so that the lever will balance?

CHAPTER XVIII.

Compound Interest and Peculiar Proportions.

1. In compound interest, interest is calculated on the *amount* of the principal at the end of each period, generally a year.

Thus the compound interest on £100 at 5 per cent. for 2 years is not £10, as it would be at simple interest, but £5 and the interest on £105 at 5 per cent. for the second year, or £5 + £5, 5s., that is, £10, 5s. in all.

2. Since it is necessary to find the *amount* at the end of each period, it is better to proceed as explained in Chapter V., viz. find the *amount* of £1 decimally for one period, and multiply the principal by it; this will give us the *amount* of the principal at the end of the first period. If, now, we consider this amount as our new principal, and multiply it again by the amount of £1 at the end of a period, we obtain the amount of the principal at the end of the second period; and for three periods we must multiply it a third time. Thus, to find the compound interest (calculated annually) on £1000 at 4 per cent. for 3 years.

Amount of £1 at 4 per cent. for 1 year is 1·04.

```
 £1000
  1·04
 ------
 £1040      Amount at end of first year.
  1·04
 ------
  4160
 1040
 -------
£1081·60    Amount at end of second year.
   1·04
 -------
 432640
108160
 ---------
£1124·8640  Amount at end of third year.
```

Therefore the interest is this amount less £1000, or £124·864$\cancel{0}$

20

17·$\cancel{4}$80s. 2s.

12

5·$\cancel{7}\cancel{6}$d. 3·36d. Ans. £124, 17s. $3\frac{9}{25}$d.

3. The above is long and tedious; it may be slightly shortened thus. To take another example—

Find the compound interest on £2504, 15s. 6d. for 3 years 4 months at 3 per cent.

First express the principal as pounds decimally thus—

12) 6·

20)15·5

£2504·775

The amount of £1 for 1 year at 4 per cent. is £1·04, and the amount of £1 for 4 months at 4 per cent. is £1·01.

We therefore have to multiply £2504·775 three times by 1·04, and this result by 1·01.

To save writing down the product of £2504·775 × 1, which, of course, is the same over and over again, let us put the multiplier to the right and the product of the principal by the ·04 in its proper relative place, so as to add it to the principal.

£2504·775 × 1·04
100 19100

£2604·96600 × 1·04
104 19864

£2709·16464 × 1·04
108·3665856

£2817·5312256 × 1·01
28·175312256

£2845·706537856 Amount at end of 3 yrs. 4 mths.
2504·775

= £340·931537856 compound interest for 3 years 4 months, or £340, 18s. $7\frac{1}{2}$d. and a small fraction more.

Since £$\frac{1}{10000}$ is less than the $\frac{1}{10}$ of a farthing, we need not keep more than 4 places to obtain the interest correct to farthings, and 3 places to obtain it correct to pence.

4. Now, if we look at this we shall notice that we have multiplied the principal by $1·04 \times 1·04 \times 1·04 \times 1·01$ to obtain the amount, and that the interest is $A - P$,

$$\text{or } P \times (1·04^3 \times 1·01) - P,$$
$$\text{or } P\{1·04^3 \times 1·01 - 1\},$$

writing $1·04^3$ for $1·04 \times 1·04 \times 1·04$, which is now an acknowledged arithmetical symbol in all examinations.

5. Arithmetic teaches us to find some of the elements of this statement if we know the others, but not all. For example—

(1.) If we know the principal P and the amount A, we can, of course, immediately find the interest.

(2.) If we know the principal and the interest and the time, we can find the rate, but this involves a knowledge of evolution, that is, extracting the square, cube, and other roots.

In the next Part we shall explain how to find *any* root. We must therefore defer these problems until we have discussed the question of evolution.

(3.) To find the number of years or periods, we have to find the number which appears as an index, as it is called, over the amount of £1 for one period at the given rate per cent., which in our example is 3 (see $1·04^3$). To find this 3 except by trial we must understand the use of logarithms, which could only be discussed to a certain extent in an Arithmetic, though they are generally included in the best text-books.

(4.) If I know the amount and the rate and time, I can immediately find the principal by dividing the amount by the amount of £1, at the end of a period at the given rate, as many times as there are periods.

6. The comparison between the simple and compound interests, or rather amounts, when all the elements are the same, is a subject that cannot be discussed without the use of symbols and a knowledge of expansions, which is beyond the region of arithmetic.

7. If 3 numbers are so connected together, as 4, 8, 16, where 4 contains 8 as often as 8 contains 16, they are said to be in continued proportion, but may be made into a simple proportion by writing the middle term twice.

$$4 : 8 = 8 : 16, \text{ whence } 4 \times 16 = 8 \times 8, \text{ or } 8^2.$$

If 4, 8, 16 are in continued proportion, it will be found that 4 contains 16 the square of the number of times that 4 contains 8, viz. $\frac{1}{4}$, which is $\frac{1}{2} \times \frac{1}{2}$, or $(\frac{1}{2})^2$.

8. In Nature and Geometry many quantities are connected together, not in a direct or inverse proportion, but directly or indirectly, as the squares or other powers of the quantities. For instance, it is found that quantities attract one another with a force which varies inversely as the squares of the distances—that is, if two bodies at any distance attract one another with a certain force, those at twice the distance will attract one another with a force $\frac{1}{4}$ as great as before.

Again, in Geometry it can be proved that triangles of the same shape are to one another as the squares of their corresponding sides.

9. In commerce we also find price and weight connected in this way. Diamonds and other precious stones are so much more valuable when they are large than when they are small, that it has been agreed to let the value of two diamonds (for instance) be to one another as the squares of the weights of the stones themselves.

Rubies, I believe, vary in price as the cubes of their weight —that is, a stone of any weight is only $\frac{1}{8}$ the value of a ruby twice as great.

Examination and Examples.

1. What would be the multiplier by which we change the principal into its amount at 6 per cent. per annum for any given number of years?

2. Find the value of £250 at 6 per cent. 2 years hence.

3. Find the compound interest on £250 at 6 per cent. *2 years hence.*

4. Find the compound interest of £270 in 2 years at 3 per cent.

5. Find the compound interest on £1397, 11s. 3d. for 3 years at 4 per cent.

6. Of what debt, due 2 years hence, at 4 per cent. compound interest, is £1397, 11s. 3d. the present value?

7. Of what sum of money is £1572, 1s. 4d. the value 3 years hence at 4 per cent., interest calculated annually?

8. What will be the amount of £512 for 3 years at $6\frac{1}{4}$ per cent.?

9. Supposing you knew the amount of £512 for 2 years and 3 years, how would you find the rate per cent.?

10. Find the amount of £2500 in 3 years at $4\frac{1}{2}$ per cent. compound interest.

11. Find the present value of £2852, 18s. $3\frac{27}{40}$d., due 18 months hence, at 9 per cent. compound interest, interest payable half-yearly.

12. What sum of money will amount to £699, 13s. $2\frac{2}{5}$d. in 2 years, reckoning compound interest for one year at 4 per cent., and for the other at $3\frac{1}{2}$ per cent.?

13. Find the compound interest on £1272 for 18 months at $9\frac{1}{2}$ per cent., the interest being calculated half-yearly.

14. Find the compound interest on £1000 for 2 years at $2\frac{1}{2}$ per cent.

15. Find the amount of £1200 at the end of 4 years, reckoning compound interest at 5 per cent. per annum.

16. Find the compound interest on £2000 for 3 years at $7\frac{1}{2}$ per cent.

17. Find the difference between the simple and compound interest on £2718, 15s. for 2 years at 3 per cent.

18. If you knew this difference, *but not the interests*, and all other elements but one, what single element could you find?

19. What sum of money will amount to £109, 5s. $10\frac{4}{5}$d. in 1 year 9 months at 5 per cent. per annum?

20. Given that the principal is £104, and the amount at some future time £109, 5s. $10\frac{4}{5}$d., and the rate 5 per cent. If I know that the time is between 1 and 2 years, find it.

21. £936, 13s. 4d. will amount to £1157, 7s. $4\frac{1}{2}$d. in a time greater than 4 years and less than 5, at $4\frac{5}{8}$ per cent. Find the fraction of a year.

22. Given that the answer to last question is $\frac{7}{8}$, find the rate per cent.

23. What will a debt of £4250 amount to if it be left standing for $2\frac{1}{2}$ years at 5 per cent. per annum, compound interest?

24. Compare the simple and compound interest on £21, 10s. at the end of 4 years, reckoning money at 5 per cent. per annum.

25. Find the discount on £5191, 13s. $7\frac{1}{5}$d. due 2 years hence, 4 per cent. compound interest.

26. Compare the simple and compound interest on £17, 10s. at the end of 3 years at 3 per cent. per annum.

27. What sum of money will amount in 3 years, at $2\frac{1}{2}$ per cent. per annum, to £134, 12s. $2\frac{23}{32}$d.?

28. Find the difference between the simple and compound interest on £119 at the end of 3 years at 4 per cent. per annum.

29. Show that this difference is the amount of the interest for 1 year at the end of 2 years at the same rate.

30. Find the amount of £260, 10s. in 3 years at $5\frac{1}{2}$ per cent. per annum, compound interest.

31. Find the compound interest of £112, 10s. for 3 years at $3\frac{1}{3}$ per cent. per annum.

32. How much will £725 amount to in 4 years at 5 per cent. compound interest?

33. What sum will amount to £1591, 13s. 2·16d. in 3 years, compound interest, the interest of the first, second, and third years being 3, 2, and 1 per cent. respectively?

34. Given the percentage of the first and third year, how would you find it for the second year?

35. Find the difference between the simple and compound interest on £383, 5s. 8d. in $2\frac{1}{2}$ years at $3\frac{3}{4}$ per cent.

36. Find the compound interest on £1000 for 4 years at 4, 3, 2, 1 per cent. the first, second, third, and fourth year respectively.

37. Give yourself the answer to 36, and find the rate of interest for third year, knowing that of the other years.

38. If a diamond weighing 3 carats be worth £10, what is *the value* of one weighing 6 carats?

39. Show that the following question states the law mentioned in paragraph 9.

If the value of each carat of a diamond increases as its weight, what is the value of a diamond of 200 carats, if one of 6 be worth £9.

40. Rubies increase in price as the cubes of their weight. If a ruby weighing 2 carats be worth £8, what is the value of one weighing 3 carats?

41. The corresponding sides of 2 triangles of exactly the same shape are 5 ft. and 8 ft. Compare their areas.

42. If the smaller triangle contain 20 superficial feet, how many does the larger contain?

43. The solid contents of cubes vary as the cubes of their edges.

If the edges of 2 cubes be as 8 to 11, compare their solid contents.

44. If the larger cube in question 43 contain 363 cub. ft., how many does the smaller contain?

45. The areas of circles vary as the squares of their diameters. Compare the areas of two circles whose diameters are as 3 : 4.

46. If the smaller circle in question 45 contain 27 superficial ft., how many would the larger contain?

47. If I make a circular plate whose diameter is 5 ft. into two circular plates, one of whose diameter is 4 ft., find that of the other.

48. The solid contents of a sphere depend on the cube of its diameter.

What is the ratio between the solid contents of two spheres whose diameters are 3 : 4?

49. If the small sphere contain 270 cub. ft., how many does the large contain?

50. Two triangles of the same shape are to one another as 64 : 100; one of the sides of the former is 10 ft., find the length of the corresponding side in the other.

CHAPTER XIX.

Proportional Parts—Partnerships—Averages—Unequal Divisions—Chain Rule—International Money Changes.

1. We explained in Chapter VII. how to divide a number into two parts, so that the parts should bear a given ratio to each other.

2. What was there said of two parts might be said of three or more. For instance, if I want to divide 100 units into three parts in the ratio of 2 : 3 : 5, which means that the first is to be $\frac{2}{3}$ of the second and $\frac{2}{5}$ of the third, or that for every two units the first has the second has 3 and the third 5. Supposing I take 10 of these units to be divided, I can give 2 of them to one, 3 to the second, and the remaining 5 to the third; hence as many tens as there are so many twos will the first have, so many threes the second, and so many fives the third. In 100 there are 10 tens, hence the first has 2 tens, the second 3 tens, and the third 5 tens.

3. Again, to divide a quantity, say 62, into three parts in the proportion of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, here again we must add $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ together, or $\frac{15+10+6}{2.3.5}$, and divide 62 by this fraction, which gives 60.

Therefore the first part is $\frac{1}{2}$ of 60, or 30; the second, $\frac{1}{3}$ of 60, or 20; and the third, $\frac{1}{5}$ of 60, or 12.

4. To find the average between a number of unequal quantities of the same kind. If I earn 4s. one day, and 5s. a second, and 6s. a third, I earn in the three days 15s., which is the same as if I earned 5s. each of the three days. My earnings, then, average 5s. a day.

5. To find the average price per acre. If 4 ac. 2 ro. are let for £10, 5 ac. 1 ro. for £12, and 6 ac. 1 ro. of inferior land for only £4 an acre.

Here I have 16 acres for which £16 is paid, hence the average value of an acre is £1.

6. I can, of course, introduce proportional parts into questions of this kind; *e.g.* if I hire 3 lots of land whose acreages are as 4, 5, 6, for which I pay per acre shillings in the proportion of 8, 6, 5 respectively, how many shillings do I pay on the average for every acre, if I pay 120s. for each acre of the first lot?

I may take the numbers 4, 5, 6, as well as any others in the same proportion, as I only want to find the price per acre, and I may take at first the rent of each acre of the first lot as 8s., of the second 6s., of the third 5s.

Then the lots cost me each year 32s. + 30s. + 30s., or 92s.; and as there are 15 acres, each acre will cost me $\frac{92}{15}$, or $6\frac{2}{15}$, supposing that the rent of the first kind were 8s. an acre; but it is 120s. an acre, hence the average price will be $6\frac{2}{15} \times \frac{120}{8}$, or 92s.

7. Questions on Partnership can be, and generally are, solved in this way.

If A have £1200 in a business, and B 2000, and C 4000, it is plain that C ought to have a larger share of the profits than A or B.

Let the profits be £1440, to be divided amongst them according to the moneys they have in the business. We have then to divide £1440 amongst 3 persons in the proportion of 1200, 2000, 4000; the proportion between these, then, will remain the same if multiplied or divided by the same number. Let us divide them by their G. C. M. 400, and thus reduce them to 3, 5, 8. Then we have to find $\frac{3}{16}$ of £1440 for A (since $3 + 5 + 8 = 16$), $\frac{5}{16}$ of £1440 for B, and $\frac{8}{16}$ of £1440 for C.

8. If they have not their money in the business the same time, we must introduce the condition of time as well as money.

Since to have £12 for 1 year is the same as having £1 for 1 month, we can easily reduce them all to the same time, and proceed as above.

tions, are worth 3 waggons, I do just as I should do in reducing tons to some smaller denomination.

Thus $3 \times 2 \times \frac{17}{3} \times \frac{11}{2} \times \frac{7}{5} \times \frac{5}{2}$ would give us the number of fowls. And as it is possible to reduce tons, cwts., qrs., etc., to some smaller denomination by adding in the cwts., qrs., etc., as we arrive at the different denominations in the process, so is it possible in reductions of this kind; thus—

How many fowls are equal to

1 waggon, 1 horse, 5 pigs, 3 geese.
2

3 horses.
$\frac{17}{3}$

22 pigs.
$\frac{11}{2}$

121 turkeys.
$\frac{7}{5}$

$172\frac{2}{5}$ geese.
$\frac{5}{2}$

431 fowls.

Similarly we could have reduced by division fowls back to waggons, horses, etc.

There is nothing more absurd in this than in any reduction. These values are not constant, as 20s. being equal to £1; but if the value of 7 geese is equal to 5 turkeys, then the value $\frac{7}{5}$ geese = the value of 1 turkey, and to reduce the values of any number of geese to that of turkeys we must multiply the number of geese by $\frac{7}{5}$, just as to change pounds into shillings we multiply by 20.

0. If we know the result and all the other connections een the different values but one, we could, of course, find connection or ratio by reducing the greater denominations to the higher of the two denominations of which we

wish to find the ratio, and reducing the lower denominations up to the lower denomination of the two to be compared, when the comparison of these numbers will give us the connection between these quantities.

11. Students are never supposed to know the connection between the moneys of different nations; but in all problems of this kind the relative values are given. Knowing them, all questions depending on them can be solved as the question in 6.

Thus, American dollars are worth 4s. 2d., an Austrian florin 1s. 11d., and a Belgian franc $9\frac{1}{2}$d., a franc being worth 100 centimes, and a dollar 100 cents.

Supposing I want to pay a bill of 4 dollars 50 cents, or \$4·50, with Austrian florins, and give 10 Austrian florins, how many Belgian centimes should I receive in change?

I give 10 florins, or $\frac{23}{12} \times 10$s., to pay $\frac{9}{2} \times \frac{25}{6}$s., and this is too much by $\frac{5}{12}$s., or 5d. A Belgian cent is $\frac{19}{200}$d., and $5 \div \frac{19}{200} = \frac{2000}{19}$, or $52\frac{12}{19}$ Belgian centimes.

12. In the course of the examples, several foreign coins with their English equivalents will be given, which, as I said before, need not be learned.

13. Supposing I know that a Holland florin is worth 1s. 8d. and a Swedish kronor is worth 1s. $1\frac{1}{4}$d., I can form a table as in paragraph 9, and immediately reduce as follows:—

A Holland florin is worth 20d., $\frac{1}{20}$ H. F. = 1d.

1s. $4\frac{1}{2}$d. is worth 1 Swedish kronor, $\frac{33}{2}$d. = 1 kronor.

Hence to change kronors to Holland florins I must multiply the number of kronors by $\frac{33}{2}$ to reduce them to pence, and this number by $\frac{1}{20}$ to reduce them to Holland florins.

E.g. reduce 80 Swedish kronors to Holland florins.

Ans. $80 \times \frac{1}{20} \times \frac{33}{2} = 66$ Swedish kronors.

14. Examples will be given in which certain coins are not of their nominal value; thus a German mark is worth 1s., but a 20-mark piece is only worth 19s. 7d. What then do I lose by paying a bill of 1000 marks in 20-mark pieces? It is evident I lose 5d. on every 20 marks, therefore on 1000 marks I lose 50×5d. or 250d., or $22\frac{10}{12}$ marks (a mark is worth 100 pfennings), therefore I lose 22 marks $83\frac{1}{3}$ pfennings.

EXAMINATION AND EXAMPLES.

1. Divide 100 into parts in the proportion of 8, 5, 7.

2. Divide 1000 into 4 parts in proportion to $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2.

3. Divide 350 into 3 parts in the proportion of 2, 112, and 103.

4. A, B, and C are in partnership; A has £1000 in the business, B £2000, and C £3000. Divide £600 fairly amongst them.

5. In question 4, if A's share were in the business 7 months, B's 4 months, and C's share 3 months, what ought each to receive?

6. A, B, and C are in business together. A and B have each £1000. At the end of the year they have £300 to divide, of which A and B have £60 each. How much money has C in the business?

7. A, B, and C are in business together, and deposited equal sums of money, but A did not bring in his money till March 1, whereas B had had his money in all the year. When did C bring in his money if he received £100 out of £540 to be divided?

8. If A received £100 more than C out of profits, when A, B, and C were in partnership together with equal sums of money, but for unequal times; if B received £240 for 12 months, and C's money was in the business 5 months, how long was A's money in?

9. A put £100 into a business on Jan. 1, B put £200 on Feb. 9, and C £300 on Feb. 29, and on April 9 they divide £680. What ought each to receive?

10. Give yourself the answer of 9 and all the elements except the time A's money was in the business, and find it.

11. Give yourself the answer of 9 and all the elements except the amount of B's capital employed, and find it.

12. A German 20-mark piece is worth £·979; find to the nearest farthing the value in English money of 3725·39 marks.

13. If a metre is 39·3708 inches, find how many metres there are in a mile, neglecting decimal parts of a metre.

14. A clock loses at the rate of 8·5″ an hour when the fire

is alight, and gains at the rate of 5·1″ an hour when the fire is not burning, but on the whole it neither loses nor gains. How long in the 24 hours is the fire burning?

15. An Egyptian piastre is $2\frac{1}{2}$d., but a gold 50-piastre piece is only worth 10s. $2\frac{1}{2}$d. How much do I lose by paying 1000 piastres in gold piastres?

16. An English sovereign will pay 98 piastres. What is the advantage of paying 1078 piastres with English gold?

17. How many Egyptian gold 50-piastre pieces ought I to receive for 2000 sovereigns?

18. A, B, C, and D have in a business £1450 in the proportion of 2, $3\frac{1}{2}$, 4, and 5, for times in the proportion of 3, 4, 5, 6. If out of £700 profits C has £300, what ought A to receive?

19. A, B, C, and D are in partnership. D has £100 more in the business than C, and C £50 more than B, and B £150 more than A. Of the profits, if D receive £100 more than C, and C £60 more than B, and B £80 more than A, find the times which B and D had their moneys in the business, if it be known that A and C had theirs three and five months respectively, and that they received £60 and £200 respectively as their shares.

20. If A have £200 more in the business than B, but B has his five months and A only four, and A receive £30 more than B out of profits worth £210; find the amount invested by each.

21. Pay £35, 6s. 5d. with German 20-mark pieces (19s. 7d.) as far as you can, and the balance in pfennings (100 pfennings = 1 mark = 1s.).

22. In China, 10 mace = 1 tael = 6s. 8d., and a China dollar is worth 4s. 6d. Convert 100 China dollars into taels and maces.

23. Pay a bill of 1900 piastres (see 15) with Italian lira ($9\frac{1}{2}$d.).

24. Pay a bill of 100 Swedish kronors (1s. $1\frac{1}{4}$d.) with Holland florins (1s. 8d.).

25. Of how much is a man defrauded if he be paid a bill of 1000 French francs (25 = £1) with Belgian francs ($9\frac{1}{2}$d.)?

26. Pay a bill of 530 marks (1s.) with Swedish kronors (1s. $1\frac{1}{4}$d.).

27. Pay a bill of £50, 10s. with an equal number of marks and Swedish kronors.

28. Pay a bill of £78, 16s. 2d. with an equal number of sovereigns, shillings, Chinese dollars (4s. 6d.), and Chinese taels (6s. 8d.).

29. An American dollar is worth 4s. 2d., but an English shilling is often taken for 25 cents (1 dollar = 100 cents). How much should I gain by passing £100 in silver shillings? Express your answer in dollars and cents.

30. A Greek drachma (= 100 lepta) = a franc ($9\frac{1}{2}$d.). Pay \$55·10 with drachmas.

31. Find a constant multiplier which will convert American dollars into English shillings or German marks, and also into Swedish kronors.

32. If 4 geese are worth 7 fowls, and 14 fowls are worth 3 pigs, and 11 pigs are worth 4 oxen, and 5 oxen are worth 3 horses, how many geese are worth 9 horses?

33. Given the data of 32 (except that 11 pigs are worth 4 oxen) with the fact that 110 geese are worth 9 horses, and find the relative value of a pig and an ox.

34. Taking relative values from question 33, how many geese are worth 4 horses, 2 oxen, and 1 goose?

35. Give yourself the result of 34 (viz. 89 geese) and all the data except the connection between pigs and fowls, and find it.

36. What is the average daily attendance at a school, if there are present on the Monday, Tuesday, etc., the following numbers of children, 43, 39, 37, 51, and 50?

37. Give yourself the average 44, and other days' attendance except Tuesdays, and find it.

38. If I purchase 11 eggs for 9d., 12 eggs for 8d., and 13 eggs for 7d., what is the average number I can buy for 2d.?

39. If three for 2d. is the average price of a quantity of eggs, of which one-third cost me $\frac{9}{11}$d. each, another third $\frac{2}{3}$ of a penny each, what did I pay for each of the remaining eggs?

40. I buy three parcels of eggs, whose numbers are as 5, 7, 9, and their prices per dozen as 8, 6, 4. How must I sell them so as neither to win or lose?

41. If in 40 the first parcel contains 3780 eggs, and costs me 15 guineas, what do the others cost me?

42. Give yourself that I paid altogether 45 guineas for the three parcels. Find what I paid for each.

43. If a clock loses per day $1\frac{1}{2}$ seconds for the four summer months, and gains $1\frac{1}{4}$ seconds for the four winter months, what is the average loss per day?

44. An estate is divided into three portions of 250 ac., 62 ac. 2 ro., and 19 ac. 1 ro. 20 po. These portions are let at £1, 5s. 4d., £1, 1s. 8d., and £3 per acre respectively. At what uniform rent per acre might the whole estate be let so as to bring in the same rental?

45. Give yourself that the average price is £1, 6s. 8d. and all the other data except the rent of the plot of 19 ac. 1 ro. 20 po., and find it.

46. Divide 45 into three parts, so that the smallest is two less and the greatest two greater than the third.

47. Men, women, and children, whose comparative number are as 3, 4, 5, and their wages as 5, 4, 3, earn altogether £460. What do the women earn?

48. Given that there are 48 women, find what each child earns.

49. If some men, women, and children, whose comparative numbers are as 4, 5, 6, and their weekly wages as 5, 4, 3, are employed for three weeks, two weeks, and one week respectively, and altogether earn £2360, find what the children earned in their week's work.

50. Given that there were 1080 children employed, find each man's weekly wages.

CHAPTER XX.

Stock, Shares, Consols, and Miscellaneous Remarks.

1. When a nation requires money for any special object, as to pay some heavy sum to another nation, or the expenses of a war, and does not care to overburden the existing generation with excessive taxation, it borrows money—often in small sums—from its own inhabitants or those of some other nation.

2. To do this, let us imagine, it prepares documents promising to pay to their possessor twice a year some given sum These documents are generally nominally worth £100, but, according as money is valuable or otherwise, they decrease or increase in value. Supposing these documents are engagements to pay £1, 10s. twice a year, that is, £3 each year, they are called the 3 per cents., because they are nominally worth £100, and give £3 each every year.

3. Looked at in this light, the buying or selling of these documents is as easy as buying or selling any other articles of commerce.

4. If I own 200 of these documents I am said to own 200×100, or £20,000 stock. If the price of them be 93, I only pay £18,600 for them, just as I should pay £18,600 for 260 horses at £93 each.

5. If I own £20,000 stock (as it is called) in the 3 per cents., considering, as the name implies, that each £100 document gives me £3 per annum, and I have 200 of them, my income will be £3×200, or £600 per annum.

6. Supposing I invest in the 3 per cents. £18,600 when they are selling at 93, what is my income? This is just the same question as supposing a horse is worth £3 a year to me, what shall I receive each year for the horses I can buy for £18,600 at £93 each? The answer, of course, is £$\frac{18600}{93} \times 3$.

7. If I buy a document which entitles me to £5 each year for £110, of course I receive £5 for use of £110, or $\frac{5}{110}$ of

the money I invest. What is this per cent? This can be found by the unitary method; thus—

If £110 give me £5
" £1 " $\frac{5}{110}$
and £100 " $\frac{5 \times 100}{110}$, or $4\frac{6}{11}$;

or it can be found by proportion—

110 : 100 = 5 : rate per cent. required,

which gives, as before, $\frac{100 \times 5}{110}$, or $4\frac{6}{11}$;

or it can be found immediately by the fractional method thus—

$100 \times \frac{5}{110}$, or $4\frac{6}{11}$.

8. The advantage or otherwise of changing stock can be found precisely as that of changing other articles of commerce. For instance, I sell £26,000 4 per cents. when they are at 98, and with the proceeds purchase 5 per cents at 104. How much less stock have I than before? '4 per cents.' and '5 per cents.' are here simply denominations like cwts. and qrs., and the numbers 4 and 5 (unless income is asked for) do not enter into our calculations at all. To work the question step by step.

£26,000 stock means 260 documents. I receive for them £260×98, and with this money I buy other documents worth £104 each; hence I purchase $\frac{260 \times 98}{104}$ such documents, and these are nominally worth £100 apiece, therefore my new stock is £$\frac{260 \times 98 \times 100}{104}$, or £24,500. Now, though I may have less stock I may have more income, because these latter documents give me £2, 10s. each half-year, and the former only £2.

My original annual income is—

£$\frac{26000}{100} \times 4$, or £1040.

My new income is—

£$\frac{26000 \times 98 \times 5}{100 \times 104}$ = £1225.

Hence I have improved my income by no less than £185.

Now let us compare these expressions for the different incomes—

$$\frac{26000}{100} \times 4 : \frac{26000}{100} \times \frac{98}{104} \times 5 = 1040 : 1225.$$

Divide the terms of the first ratio by $\frac{26000}{100 \times 98}$ and we get

$$\frac{4}{98} : \frac{5}{104} = 1040 : 1225.$$

Now, if I know any five of these six elements I can find the other, viz. the 4 or 5 (called rates), or 98 or 104 (called prices), or either of the two incomes, without knowing the principal at all.

The $\frac{4}{98}$ may need a word of explanation.

If I can only procure £98 for that which will give me £4 a year, I receive $\frac{4}{98}$ of my money in the form of income.

If I know the difference between the two incomes, viz. £85, I can find the incomes, and from either of them the principal.

If I know the difference between the income, say £185, what I have to do is to form a fraction whose terms differ by this difference (£185) equal to the fraction.

$$\frac{\frac{4}{98}}{\frac{5}{104}}, \text{ or } \frac{4 \times 104}{5 \times 98}, \text{ or } \frac{208}{245},$$

$$\frac{208}{245 - 208} = \frac{\text{some numerator}}{185},$$

$$\frac{208}{37} = \frac{208}{\frac{37}{1}} = \frac{\frac{208}{37} \times 185}{185} = \frac{1040}{185},$$

$$\frac{208}{208 + 37} = \frac{1040}{1040 + 185};$$

$$\therefore \frac{208}{245}, \text{ or } \frac{\frac{4}{98}}{\frac{5}{104}} = \frac{1040}{1225}.$$

Hence the incomes are £1040 for the 4 per cents. when worth 98, and £1225 for the 5 per cents. when worth 104.

9. If I know the principal invested (not the stock purchased) and the difference of the incomes and one of the rates with its price, I can find both the other rate and its price. To take the same example. If £185 is the difference of income

on all the principal, then $\frac{185}{25480}$ is the difference on each £1 of the principal.

But the income from the better (say) investment for each £ is £$\frac{5}{104}$. Then I have to find what fraction is to be subtracted from £$\frac{5}{104}$ so as to leave $\frac{185}{25480}$, which I find by subtracting $\frac{185}{25480}$ from $\frac{5}{104}$; thus $\frac{1225-185}{4\times 49\times 13\times 10}$,

$$\text{or } \frac{1040}{4.49.13.10}, \text{ or } \frac{4}{98};$$

hence the other investment must be in the 4 per cents. when they are at 98.

10. In some Arithmetics you find problems of this kind. (We will take the numbers from the last, and ask it in a different way.)

I invest £26,000 in the 4 per cents. when they are at a certain price, but on their falling £2 a document I purchase 5 per cents. at 104, and thereby improve my income by £185. At what price did I originally invest?

If I use some symbol as the original price of the stock or the number of shares I have, it is not difficult; but this, I maintain, is not arithmetic. Not knowing either the price I invested in, or the number of documents I had, I cannot tell what my original income was, nor can I tell my real income per £ without I know for what I receive the £4. It is usual to call the fall £2 per cent. (in this case it actually is 2 per cent.), and if it were so the price could be found; but supposing I had originally invested at 98 and they had dropped to 96, this fall is more than 2 per cent., being really 2 per 98, or $2\frac{2}{49}$ per cent.

Since this £2 is really a fall of 2 per cent., then I am really receiving an income from an investment of $\frac{98}{100}$ of £26,000, or £25,480; hence the difference on every £ is £$\frac{185}{25480}$, and the new investment gives me £$\frac{5}{104}$; hence the former one, as before, gives me £$\frac{4}{98}$; hence I originally purchased at 98 + 2 or 100, or, as it is called, at par.

11. We will take another example. A person invests £3080 in the 4 per cents. at a certain price. On the stock rising

$2\frac{3}{11}$ per cent. of the original price (not £$2\frac{3}{11}$ for each £100 of stock) he sells out and purchases 5 per cents. at 105, thereby increasing his income by £10. Find the price at which he originally bought.

Note.—When stock can be sold at a certain price, your true percentage is what you receive on what you can get for your stock, not on what you paid for it. If a man has a garden on a lease for which he pays £5 a year, and which he could if he chose sublet for £1000 a year, his garden rent is £1000 per annum, not £5. When, then, the stock rose $2\frac{3}{11}$ per cent. on its original value he was receiving his £4 for $\frac{102\frac{3}{11}}{100}$ of £3080; hence his capital was really this sum, and not £3080, viz. £3150. Hence the difference on each £1 is £$\frac{10}{3150}$, and as before £$\frac{5}{100} - \frac{10}{3150}$, or $\frac{150-10}{3.105.10}$, or $\frac{140}{3.105.10}$, or $\frac{2}{45}$, or $\frac{4}{90}$; he therefore sold out at 90, and 90 is $\frac{102\frac{3}{11}}{100}$ of 88; hence he originally purchased at 88.

12. I am sorry to be obliged to acknowledge that I cannot solve—without the use of symbols—questions of this kind, in which the actual variation, and not the ratio of variation, in price is given. Here is one done, but it is not arithmetic.

I invest £3600 in the $4\frac{1}{2}$ per cents. at a certain price; on the stock rising £5 for every £100 document (this may not be 5 per cent., though usually so described), I sell, and with the proceeds purchase 5 per cents. at par, thereby making an improvement in my income of £10. At what price did I originally buy?

If we use P as the original price, then my original income must be considered to be derived from $\frac{3600\ (P+5)}{P}$, and the change in income on each £$1 = \frac{10 \cdot P}{3600\ (P+5)}$, or $\frac{P}{360\ (P+5)}$; but $\frac{5}{100}$ is my new income on each £1, hence $\frac{5}{100}$

$-\dfrac{P}{360\,(P+5)}$ will give me the true income on each £1 of the original investment, and this is $\dfrac{9}{(P+5)\,2}$; and hence to find P we have

$$\frac{5}{100}-\frac{9}{2\,(P+5)}=\frac{P}{360\,(P+5)},$$

$$\text{or } \frac{10\,P+50-900}{200\,(P+5)}=\frac{P}{360\,(P+5)}$$

$$\begin{aligned}\text{or } 90\,P+450-8100&=5\,P,\\ \text{or } 85\,P&=7650,\\ P&=90.\end{aligned}$$

13. The difficulty of these examples depends, as stated above, on the fact that the money changes in income-producing capabilities as the price of the stock varies.

When we talk in history of a penny being worth as much as a shilling now, we express an idea of the same class.

14. There is a seeming difficulty in percentage questions that I must add a line about before concluding this Part. Had I not seen the mistake printed and discussed as a difficulty, it would not have occurred to me as one that could have been made.

The author of this book seemed to think it possible to ask one question and get two answers equally correct, and this without the use of symbols.

It was, as far as I remember, a question something like this. A man purchases eggs at 5d. a dozen and sells them at 4s. a hundred, how much does he gain per cent.?

(1.) 100 eggs cost him $8\frac{1}{3}\times 5=41\frac{2}{3}$d., for which he receives 48d.; hence his gain per cent. is $6\frac{1}{3}$.

(2.) 100d. would buy 20 dozen, for which he would receive $115\frac{1}{5}$d.; hence his gain is $15\frac{1}{5}$ per cent.

(3.) A third result was obtained (but I forget how) of $31\frac{2}{3}$, and I am not sure that there was not even a fourth result obtained.

No. 2 is right. 100d. worth of eggs are sold for $115\frac{1}{5}$d.; *therefore* the profit per cent. is $15\frac{1}{5}$.

In No. 1 it is quite true there is a profit of $6\frac{1}{8}$d. made in selling 100 eggs, but not a profit of $6\frac{1}{8}$ per cent.; for 100 eggs only cost $41\frac{2}{3}$d. Therefore this $6\frac{1}{8}$ profit is not the profit on 100d., or a percentage profit, but the profit on $41\frac{2}{3}$d.

Again, in No. 3 the profit on 100d. worth of eggs is the price of $31\frac{2}{3}$ eggs. But this again is not the profit per cent., but the $31\frac{2}{3}$ is the profit on eggs that can be purchased for 100d.

The denominators must be the same; if our result is pence we must compare them with 100d. Again, if the result is in eggs, we must compare them with eggs. Reduced to eggs or to pence, as the case may be, all the results would be the same, viz. $15\frac{1}{8}$ per cent.

Examination and Examples.

1. What do you understand by consols being at $101\frac{1}{2}$?

2. If £25 shares are selling at £3 premium, what do you pay for 20 shares?

3. I exchange 280 £20 shares which are selling at £1 discount for some £25 shares which are selling at £3 premium. How many shares did I receive?

4. What is the difference between a nation's funded and unfunded debt?

5. Explain the expressions, '£25 shares were selling for £36 ex. dividend.'

6. What income would be obtained by investing £12,002, 12s. $4\frac{1}{2}$d. in 3 per cent. stock at 93?

7. If £12,002, 12s. $4\frac{1}{2}$d. in the 3 per cents. give an income of £387, 3s. $7\frac{1}{2}$d., at what price was the stock purchased?

8. What must be the price of a 6 per cent. stock in order that money invested in it may yield $4\frac{1}{2}$ per cent.?

9. If I sell out of a 6 per cent. stock when it is at $133\frac{1}{3}$, what per cent. does the investment pay me?

10. I sell out £5000 4 per cent. stock at 108, and with the proceeds buy 5 per cents. at 120. What is the change in my income?

11. If I sell £5000 which I had in the 4 per cents. and with the proceeds buy 5 per cents. at 108, and thereby increase my income by £25, at what price do I sell?

12. What sum must be invested in the $3\frac{1}{2}$ per cents. at 104 to obtain an income of £329?

13. In what stock do I invest £9776, when they are at 104, if I obtain an income of £329?

14. What sum must be invested in 4 per cent. Spanish stock at $63\frac{1}{2}$ to give an income of £340, 4s. 8d.? *N.B.*—Brokerage not to be reckoned.

15. Find the above if the brokerage of $\frac{1}{8}$ per cent. were reckoned.

16. If money invested in the 3 per cent. consols gives exactly 3 per cent. after paying 5d. in the pound income tax, what is the price of the consols?

17. If the brokerage on purchasing 3 per cents. is enough to deprive me of £1 of my income, what amount do I buy?

18. A person investing in the 3 per cents. receives $3\frac{1}{3}$ per cent. interest for his money. What is the price of the stock?

19. How much money must he invest so as to have an income of £204 a year?

20. What is the gain by investing £1950 at $97\frac{3}{8}$ and selling out at 104, brokerage being on each transaction $\frac{1}{8}$ per cent.?

21. If by purchasing stock at $97\frac{3}{8}+\frac{1}{8}$ for brokerage, and subsequently selling at $104+\frac{1}{8}$ for brokerage, and thereby gaining £127, 10s., what amount did I invest?

22. A man invested a sum of money in the 3 per cent. consols at 88, and at the end of $4\frac{1}{2}$ months, after receiving one half-year's dividend, sold out at $87\frac{7}{8}$. At what rate per cent. per annum did he receive interest on his capital?

23. A speculator, with private information, purchased £100,000 of Italian stock at $65\frac{3}{8}$; in the course of the day it rose $1\frac{3}{8}$ per cent. When he sold it all, what did he make by the transaction? and what did the broker receive?

24. A person having a certain sum of money to invest, finds that an investment in a railway preference 5 per cent. stock at $117\frac{1}{2}$ will yield him £29 more interest annually than an investment in the 3 per cent. consols at $92\frac{1}{4}$. How much money has he to invest?

25. A man invests £1012$\frac{2}{3}$ in the 3 per cents., but on the stock rising to 99 he sells out and reinvests in the 4 per cents. at 110, thereby increasing his income £6, 4s. At what price did he originally invest?

26. Which is the better investment—French 3 per cents. at 64$\frac{1}{2}$, or United States 5 per cents. at 102$\frac{3}{4}$? Calculate the difference of income produced by an investment of £1000 in each.

27. Compare the investments, 5 per cents. at 105, and 4 per cents. at 96, and 3 per cents. at 90.

28. £3600 of 3$\frac{1}{2}$ per cent. stock is purchased for £3519. What is the price of the stock? and what income would be thence derived?

29. If an income of £126 is derived from 3$\frac{1}{2}$ per cent. stock when at 97$\frac{3}{4}$, what amount of money do I invest?

30. A man sold £19,200 3 per cent. stock at 85, and invested the proceeds in 4 per cent. stock at 96. What was his income before and after the transaction?

31. Is the £19,200 the nominal or the money value of the stock?

32. How much money must be invested in the 3 per cents. at 87 in order to produce a net income of £295 after deducting income tax at 4d. in the pound?

33. If the net income, after deducting income tax, is £295 from investing £8700 in the 3 per cents., what is the amount of the income tax in the pound?

34. A person has a certain capital, half of which he invests in the 3 per cents. at 90 and the other half in the 5 per cents. at 110. What was his capital if his total income (from both sources) is £6883, 10s.?

35. A person has a certain capital, one-third of which he invests in the 5 per cents. at 110, another third in the 4 per cents. at par, and the remainder in the 5 per cents. at 110; altogether he receives an income of £23092$\frac{4}{11}$. What did he invest?

36. If 3 per cent. consols be at 90$\frac{5}{8}$, what sum must I invest in order to secure from them a yearly income of £470, after paying an income tax of 5d. in the pound, brokerage being at $\frac{1}{8}$ per cent.?

37. To what extent must the price be diminished that my income may remain the same if I had invested the same amount of money when the income tax was 8d. in the pound?

38. If the price of the 3 per cent. stock be 96, a person can obtain an annual income of £1 more than he can if the price be 97. How much has he to invest?

39. A man invests £6920 in the 5 per cents. when they are at 108; after 2 months he receives the half-yearly dividend; and at the end of 1 more month sells out at £109$\frac{7}{8}$. If he pay brokerage $\frac{1}{8}$ per cent. and an income tax of 5d. in the pound, what per cent. does he make per annum?

40. An investment of £7440 is made in 3 per cent. consols at 92$\frac{7}{8}$ two months before the half-yearly dividend is paid. Immediately after the receipt of the dividend the stock is sold out at 92. If a brokerage of $\frac{1}{8}$ per cent. is paid both on the purchase and sale, find the rate of interest per cent. per annum obtained by the investor on this transaction.

41. A person invests a certain sum of money in the 3 per cents. at 90, £110 more in the 4 per cents. at par, and £880 more than this in the 5 per cents. at 110, and altogether receives as income £167. How much did he invest in the 3 per cents.?

42. A man has stock in 3 per cents. which brings him £240 a year. He sells out one-fourth of the stock at 87$\frac{1}{4}$, and invests the proceeds in railway stock at 174$\frac{1}{2}$. What dividend per cent. per annum ought the railway stock to pay so that he may increase his income £40 per annum by the transaction?

43. A man has £8820, which he invests at a certain price in the 3$\frac{1}{2}$ per cents.; on the stock increasing in price to the extent of $\frac{1}{45}$ he sells out, and reinvests in the 4 per cents. at 98, thereby increasing his income by £25. At what price did he buy?

44. A certain income can be obtained by an investment in the 3$\frac{1}{4}$ per cents. at 93$\frac{3}{8}$; the same income can also be obtained by investing £1595 less in the 4$\frac{1}{4}$ per cents. at 105$\frac{5}{8}$, no brokerage being charged in either transaction. What is the income?

45. Give yourself the answer of 45, viz. £405$\frac{1}{8}$, and the other data except the price of the 3$\frac{1}{4}$ per cent. stock, and find it.

46. A person who has £10,257, 10s. 3 per cent. stock calculates that, by selling it and investing in $3\frac{1}{4}$ per cent. stock at $93\frac{1}{4}$ he can increase his annual income by £10, 9s.; but before he can effect the exchange, each stock rises $\frac{1}{4}$ per cent. By how much is his income really increased?

47. If an investor only makes $\frac{5}{17}$ per cent. more by investing in the 5 per cents. than the 4, compare their prices.

48. The difference between the incomes derived from investing a certain sum in 5 per cent. stock at 127 and $5\frac{1}{2}$ per cent. stock at 135 is £4, 14s. Find the amount invested.

49. Find the income resulting from each investment in question 48.

50. I buy eggs at 8d. a dozen and sell them at 7s. 6d. a hundred. What percentage do I make? Express your answer both in pence and eggs, and show that they are the same.

MISCELLANEOUS QUESTIONS.

I.

1. Find a ratio, the terms of which differ by 3, = 5 : 6.

2. Find a ratio, the sum of whose terms is 10, = 1 : 4.

3. If a 5d. loaf weigh 4 lbs. when wheat is at 30s., what ought it to weigh when wheat is 6s. more?

4. Find a constant multiplier which will connect the price of wheat with that of the 4 lb. loaf if 5d. will buy it when wheat is at 30s.

5. What unit for the price of the loaf could I adopt so that the number of the shillings which pays for the wheat could give me the price of the loaf?

6. Find two numbers, the first of which is $\frac{3}{7}$ of the second, and the second greater by 12 than the first.

7. Find the nearest number to 1000 which will divide by 17 and leave a remainder 1.

8. If the death-rates in a town per 1000 for 4 consecutive years are 19·01, 20·3, 19·07, and 18·62, what is the average death-rate? and if the deaths in the 2nd year were 812, what is the population?

II.

9. Divide 98 units into four parts in the proportion of 2, 3, 4, 5; and if the 3 represent 630, what is the unit employed, and what number does the 5 represent?

10. Reduce 60 miles an hour to $\frac{1}{2}$ ft. per quarter second.

11. A man is running at 6 miles an hour along a train in the opposite direction to which it is going, and meets another train on which a man is running 8 miles an hour in the same direction as the train. If the trains are going 56 and 42 miles an hour respectively, how far are the men apart at the end of 36 seconds from the moment that they passed one another?

12. Divide 37 into two parts in the ratio of 3 to 5.

13. How many more times is a farthing contained in a £ than 2 oz. in a cwt.?

14. If in goods which cost £1 a cwt. you calculated a farthing for every 2 oz., what would be the least number of ounces when this mode of calculation would make you a farthing wrong?

15. If the large wheel of a bicycle is 18 ft. 4 in. in circumference and the smaller one 4 ft. 2 in., how often do the same spots touch the ground at the same moment in 5 miles?

16. Given this 288 times, the 5 miles and the circumference of smaller wheel, find that of larger.

III.

17. There is a number which leaves the remainder 2 after it is divided by 7, 9, and 5. Find it.

18. Find the greatest number which will divide 614 and 666 so as to leave a remainder 3.

19. What is the interest on 1d. for 120 years simple interest at 4 per cent. per annum?

20. What fraction taken $\frac{3}{7}$ of a time is 2 less than $4\frac{1}{5\frac{1}{6}}$?

21. Divide 317 into three parts in the proportions of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$.

22. If a certain interest is obtained from three investments,

viz. 5 per cents., 4 per cents., and 3 per cents., if the interests are all the same, compare the sum invested.

23. If the investments are the same, compare the interests on each investment.

24. If the interests were £120 when all the investments were equal, what sum was invested?

IV.

25. State and work as a Compound Proportion question. Find at what percentage per month must I lend £50 for 6 months so as to receive £1.

26. If I receive £3 more for lending money at 5 per cent. than at 4, what amount of money do I lend?

27. What is the interest for £1 for a day at the rate of 5 per cent. for 300 days?

28. How can a bill of 1s. 1d. be paid with 4d. and 3d. pieces?

29. Divide £109 amongst 100 men, 100 women, and 100 children, so that the men have 1s. each more than each woman, and each woman $\frac{5}{8}$ of a child.

30. If A has $\frac{2}{5}$ of B, and B $\frac{5}{7}$ of C, and C $\frac{14}{23}$ of D, what has D of A?

31. If in 30 B have £100, what has A?

32. If A have $\frac{2}{5}$ as much again as B, how much less than A has B?

V.

33. If 1s. 3d. is the difference between the interest and the discount at 5 per cent. for 3 years, find the principal.

34. If £2, 10s. is the difference between investing money in the 5 per cents. at 95 and 4 per cents. at 80, find the principal.

35. Find a fraction $= \frac{3}{5}$ whose terms differ by 24.

36. Find a fraction $= \frac{4}{7}$ the sum of whose term is 33.

37. A and B can do a piece of work in 6 days, B and C

in 8 days, and C and A in 10 days; in what time could they all do it together?

38. A can give B 200 points out of 1000, B can give C 300, how many ought A to give C?

39. Having given that 125 Italian lire make 23 Roman scudi, and that 2001 Roman scudi make 12500 Austrian zwanzigers, find how many Italian lires make 50 Austrian zwanzigers.

40. What is a man's gross income if after paying income tax of 5d. in the £ he has £705?

VI.

41. At what price must the 3 per cents. be that I may make 4 per cent. of my money?

42. A man buys 100 sheep, he loses 10, and 10 he has to sell at a loss of 10s. each, and on the rest he makes a profit of 5s. each, and thereby making a profit of 5 per cent. on his transaction. What did the sheep cost him?

43. The attendance at a school is 8 per cent. less on Tuesday than on Monday, on Wednesday it is 5 per cent. more than on Monday, on Thursday 5 per cent. more, and on Friday 5 per cent. less. If there were 26 more children at school on Thursday than on Tuesday, what was the daily average attendance?

44. Change 34710 nonary to septenary.

45. Three persons, A, B, C, agree to pay their hotel bill in the proportion 4 : 5 : 6. A pays the first day's bill, which amounts to £2, 10s. 10d.; B the second, viz. £3, 12s. 2d.; C the third, of £3, 17s. How must they settle?

46. A person finds that if he invest a certain sum in railway shares, paying £6 per share when the £100 share is at 132, he will obtain £10, 16s. a year more for his money than if he invest in 3 per cent. consuls at 93. What sum has he to invest?

47. Take $\frac{1}{6}$ as often as you can from $2\frac{1}{2}$, and state what is the nature of the remainder.

48. A railway train leaves Edinburgh at 9 for London (405 miles) at 66 ft. per second. When will it meet the train which

left London at 10, travelling at the rate of 1320 yards a minute?

VII.

49. Divide £13 amongst A, B, C so that A's share is to C's as 3 to 5, and B's 2 more than A's.

50. Divide 46 into two parts so that the same figures will denote the numbers expressed in nonary and denary respectively.

51. Two sets of men, whose numbers are as 4 : 5, and who work for days whose numbers are as 5 : 6, earn the same wages. Compare their relative value as workmen.

52. Three men can earn as much as 5 women, and 8 women as much as 12 children. If in 10 days 1 man, 2 women, and 3 children earn £1, 14s. more than they would in 8 days, what are the earnings of each for a day?

53. If a man, after saving $\frac{1}{5}$ of his income, determine to reduce his expenditure 20 per cent., and thereby finds he had £100 more at the end of the year than he had before, what was his income?

54. Divide 89 into two parts, so that the same digits will express them in the quinary and senary scales respectively.

55. Obtain a constant multiplier which will change the number of cwts. of sugar into pence per lb. supposing £1 will buy a cwt.

56. Divide 37 into 3 parts in the proportions $\frac{1}{2}$, $\frac{1}{4}$, 2.

VIII.

57. A man invests half his capital in the 3 per cents. at 91, and the remaining half in 4 per cents. at par, and finds he receives £8 more income from the one investment than from the other. What was his capital?

58. Find the compound interest on £1000 for 2 years at $3\frac{1}{2}$ per cent.

59. What number must be added to $\frac{1}{6}$ so that if the sum be divided by the difference between $\frac{1}{2}$ and $\frac{1}{3}$ the quotient may be $1\frac{4}{5\frac{1}{6}}$?

60. A man purchases some 5 per cent. stock at 104 1 month before the half-yearly dividend is due, and 2 months after it is paid sells out at 106. What per cent. per annum does he make for his money?

61. If the weight of a cubic foot of water is 63·35 lbs. avoirdupois, what is the error in calculating the weight of 1000 cubic feet on the supposition that a cubic fathom (216 cubic feet) weighs 6 tons?

62. Divide 47 into three parts so that the first is to the second as 2 to 3, and the third 12 greater than the first.

63. If eggs be bought at 21 for a shilling, how many must be sold for a guinea to give a profit of $12\frac{1}{2}$ per cent.?

64. If I make 1 per cent. more by selling eggs at 10d. a score instead of 5d. a dozen, what do I buy them at?

IX.

65. What is the insurance on £12,000 at 3 per cent. if I so insure as to save the premium as well as the property in case of loss?

66. Show that it is to the advantage of the insurer that his property be lost.

67. How many more integral times is $\frac{1}{2}$ contained in $13\frac{1}{4}$ than ·16 in 3·4?

68. A garrison of 640 men has provisions for 124 days; how long will the provisions last if 200 women are taken in, after which the men's rations are reduced 20 per cent., and 5 women have served out to them the rations of 4 men?

69. How is it that the number of days is unaltered?

70. Give yourself the elements except the number of men, and find it.

71. A's income is $\frac{2}{5}$ of B's, and B saves £1 a week. If they spend the same amount of money, what is A's annual income?

72. A man walks 5 miles the first hour, he then rests for $\frac{1}{4}$ of an hour, he then walks at 4 miles an hour for $1\frac{3}{4}$ hours, but after that drops down to 3 miles for 2 hours more. What is his average pace, counting his stop?

X.

73. A person paid £18, 15s. for a year's income tax. When Government increased it to 9d. in the £ he paid £52, 10s., what was the original tax?

74. How many hours a day must 24 men work to accomplish as much in 5 days as 25 men could do in 4 days if they worked 6 hours a day?

75. If you knew that the 24 men who worked for 5 days worked 1 hour a day less than the other men who worked 4 days and did the same work, find the second number of men.

76. A man purchases £700 stock in 3 per cents. at 94½, and also invests £585 in Russian 5 per cents. at 97½. How much stock has he standing in his name?

77. If he sells out of the 3 per cents. at 95, and from the 5 per cents. at 96½, how much does he lose?

78. A, B, C are in partnership. A only had £100 in the business, but B did not come into the business with £900 till July 1, and C on November 1. If out of £200 of profits C's share is £50, what money did he bring into the business?

79. If a man walking at the rate of 4 miles an hour can travel a certain distance in 3 hrs. 25 min., in what time could he run the distance at the rate of 7 miles an hour?

80. Reduce 3·45 nonary to septenary.

XI.

81. A publican mixes 4 gallons of gin, which is worth 15s. a gallon, with 4 gallons of water and a gallon of base spirits worth 10s. What will he gain per cent. on his outlay by selling the mixture at 2s. 10d. per bottle of six to the gallon?

82. I find that by selling eggs at 6d. a dozen I make 102 per cent. more than by selling them at 8d. a score. How many eggs must I purchase so that if I sell half at 6d. a dozen and the other half at 8d. a score, I make £13, 9s. 3d.?

83. What is the net income of a man if 1d. in the income tax makes a difference of £2, 5s. in his income?

84. If £3600 be received as income for three equal investments, one in the 5 per cents., another in the 4, and the third in the 3 per cents., find the amount of each investment.

85. A farmer purchases 749 sheep, and sells 700 of them for the price he paid for the whole, and afterwards sells the remaining sheep at the same price per head as the others. Find the gain per cent.

86. A tradesman's prices are 25 per cent. above cost price. If he allow a customer 12 per cent. on his bill, what profit does he make?

87. A person buys some tea at 3s. a lb., and some at 2s. a lb., in what proportion must he mix them so that by selling his tea at 2s. 7½d. a lb. he may gain 20 per cent. on each lb. sold?

88. Divide 13 into two parts, so that 7 times the one may be greater by 6 than 10 times the other.

XII.

89. There is a number, I double it and take away 4 from it, and then divide the result by 8. This I multiply by itself and take 1 from it, and find a remainder 15. Find the original number.

90. I divide £205 amongst A, B, C, and D. A has £10 more than C, and B £15 more than D, and A's share is equal to D's. Find C's.

91. Divide 286 into three parts so that the same digits in the same order will express the three numbers in the quinary, senary, and septenary scales respectively.

92. Divide 200 into two parts so that the digits of one number expressed in the quinary are the same in a reverse order in the other expressed in the septenary.

93. Prove that 297 could not be divided into two numbers which could be expressed by the same digits in the quinary and septenary scales.

94. A hare is 200 of her own paces in front of a greyhound. If 2 of the greyhound's paces are equal to 3 of the hare's, but he only takes 3 whilst the greyhound takes 4, in how many of the hare's paces will he overtake him?

95. If by settlement a man cannot remove his money from the funds, prove that his money decreases in value as the funds rise in price.

96. A person having to pay £402, 3s. 9d. two years hence, invests a certain sum in the 3 per cents., and also an equal sum next year together with the interest already received. Supposing the price of consols to remain throughout at 96, what must be the sum invested on each occasion so that there may be just sufficient to pay the debt at the proper time?

PART III.

CHAPTER XXI.

Square Root.

1. 45^2 (read 45 square) is a short way of writing 45×45, or 2025.

2. The operation of obtaining 45×45, or 2025, from the expression 45^2 is called involution. This in arithmetic is always perfectly simple in method, but generally laborious in execution, involving more or less multiplication.

3. The reverse of this operation is called evolution.

4. $\sqrt{2025}$ (called the square root of 2025) is the number 45, because 45×45 is 2025.

5. Hence the square root of a number is that number which, multiplied by itself, will produce that number.

6. To find the square root of a number we must know the squares of the first nine numbers; these are 1, 4, 9, 16, 25, 36, 49, 64, 81.

7. Since 10 (the least number that can be expressed by two figures) square is 100, and 99 (the greatest number that can be expressed by two figures) square is 9801, we know that the square of all numbers expressed by two figures must have three or four figures in the square. Hence the square root of all numbers expressed by three or four figures have two figures in the root. Similarly the square root of all numbers expressed by five or six figures will have three figures in the root, and so on.

8. To form the square of any number—as 47. Let us write it thus: $40 + 7$, and multiply it by the number expressed in the same way; thus—

$$\begin{array}{l} 40+7 \\ 40+7 \\ \hline 1600+7\times 40 \\ \quad +7\times 40\times 49 \\ \hline 1600+2\times 7\times 40+49 \end{array} \quad \text{or} \quad \begin{array}{r} 1600 \\ 560 \\ 49 \\ \hline 2209 \end{array}$$

9. To reverse this operation and find the square root of 2209. We mark the number in pairs to see how many figures there are in the root, as explained in 7. Thus $\widehat{22}\widehat{09}$. Since 22 is greater than 16 and less than 25, we know that the root must be between 40 and 50. Hence I can immediately find the 1600, which I subtract from the 2209 thus (the ciphers being understood as in Division). Now, if I look at the involution

$$\begin{array}{r} \widehat{22}\widehat{09}(40+7 \\ 16 \quad\quad \\ \hline 80)609 \quad\quad \\ 560 \quad\quad \\ \hline 49 \quad\quad \end{array}$$

operation in 8, I see that I have left $2\times 7\times 40+49$, of which from the 40 and the remainder 609 I have to find the 7. The greater part of this is 560, of which I know the 80 or twice the 40; let me then divide the remainder 609 by 80 and see how often it will go. Thus we get 7 for the other figure, and if the remainder be 7×7, or 49, we know 47 is the square root; since the remainder after subtracting the $2\times 40\times 7$ ought to be 49. I

$$\begin{array}{r} \widehat{22}\widehat{09}(47 \\ 16 \quad \\ \hline 87)609 \quad \\ 609 \quad \\ \hline \end{array}$$

can test this just as well by inserting the 7 after the 8 and multiplying it by the 7, when I multiply the 80 as below, and writing the 40 and the 7 of the root together.

10. Let us take another example, 85849. We first find the

$$\begin{array}{r} \widehat{8}\widehat{58}\widehat{49}(293 \\ 4 \quad\quad \\ \hline 49)458 \quad\quad \\ 441 \quad\quad \\ \hline 583)179 \quad\quad \\ 179 \quad\quad \\ \hline \end{array}$$

nearest square number less than 8, viz. 4, and put down its root. Then subtracting and bringing down two figures we divide the 45 by 4; it goes 11 times, but the root cannot contain a digit greater than 9 so we insert the 9 after the 4 and in the root. We now treat the 29 just as we treated the 2, viz. double it, and divide the 179 by

it and insert the 3 both after the 58 and in the root. There being no remainder, we know that 85849 is a square number, and that 293 is its square root.

11. There is another *form* of the working, which, though not quite so short for the square root, has the advantage of being applicable to any root. The student will see, if he examine carefully, that the operation is precisely the same, and only differs in form.

o[1]	
2 the hundreds.	$\overset{\frown}{8}\overset{\frown}{58}\overset{\frown}{49}$(293
2 the hundreds.	4 the hundreds squared.
2 the hundreds.	
40 twice the hundreds.	45849 the number less the hundreds squared.
9 the tens.	[2]441.
49 twice the hundreds + the tens.	
9 the tens.	
580 twice the hundreds + twice the tens.	1749 the number (less the hundreds + tens) squared.
3	[3]1749.
583 twice the hundreds and twice the tens taken three times + the units.	

[1] This o is placed here simply to remind us that when we take down the next pair of figures we must treat the 4 as 40, which, of course, it is all along; only at first the cipher can be understood.

[2] This 441 is twice the hundreds taken nine times and the tens squared.

[3] This 1749 is twice the hundreds + tens taken three times + the units squared.

It will be noticed that there is one addition and one subtraction. We first multiply the 1 by the hundreds figure in the root, and put down product under or to the left of o, and add; multiply this sum 2 by 2, and put the 4 under the

number; this gives us the square of the hundreds to subtract. Then begin again, and multiply the 1 by the 2 and insert it under the other 2, draw a long line, add the 2 and 2 and place the cipher after it, but subtract the 4 from the 8 and bring down next two figures. Divide the 458 by 40, and, as before, we get the 9, then multiply the 1 by the 9 and insert the 9 under the 40 and add them together, and multiply this sum by 9 and put it under the 458. Then multiply the 1 a second time by 9 and place it under the 49, and draw, as before, a long line. Add the 49 and the 9 and put the cipher on, and subtract the 441 from 458. Lastly, divide the 1749 by 580, and proceed as before.

As I said before, this, for the square root, is not so short as the method in paragraph 10, but I insert it here to show its connection with the best method for finding the 3rd or cube root, and the only method ever adopted for finding the 5th or 7th or higher (properly lower) roots. We will work one more example, and we advise the student to work several.

Find the square root of 21224449.

```
1        0       21224449(4607
         4    /16
         -  4/
         4/
         4
------------------------------------
1        80       522
          6    /516
         --  6/
         86/
          6
------------------------------------
1        9200       64449
            7    /64449
         ----  7/
         9207/
```

Since 920 is not contained in 644, we know the ten figures *of the root is* 0; we therefore bring down two more figures and

add on a cipher to the 920. We have drawn lines to show where the number in the right-hand column comes from.

12. Find the square root of 3659·0401.

$$3^2 = 9 \text{ but } (3 \times \underline{10})^2 = 900$$
$$\sqrt{9} = 3 \text{ and } \sqrt{900} = 30.$$

From this we see that if I multiply the number whose square root is to be found by 100, I obtain a root 10 times as great. Similarly, if I multiply a number by 10000, I multiply the root by 100, or 1 followed by half as many ciphers as I multiplied the number.

If in 3659·0401 I move the decimal point to the end, I multiply the number by 10000, hence I multiply the root by 100; when, therefore, I have found the root of 36590401, to find that of 3659·0401 I must divide the root of 36590401 by 100, or move the decimal point two places to the left from the right of the units, where it is understood to be.

```
       36590401(6049
       36
       ----
  1204)5904
       4816
       ----
 12089)108801
       198801
       ------
       .....
```

The root then of 3659·0401 is 60·49.

13. Find the square of 14·4.

$$14{\cdot}4 = \frac{144}{10} = \frac{12}{\text{square root of } 10}.$$

The square root of 10 is some number between 3 and 4. Being greater than 9 and less than 16, we can multiply 10 by 1 followed by as many *pairs* of ciphers as we please, if we will divide the root found by 1 followed by as many ciphers as we multiplied the 10 by the 1 followed by double the number of

ciphers. Let us multiply the 10 by 1000000 and divide the root found by 1000; thus—

$$\begin{array}{l}
\quad\ 10000000(3163,\text{ etc.}\\
\quad\ \ 9\\
\hline
61)100\\
\quad\ \ 61\\
\hline
626)3900\\
\quad\ \ 3756\\
\hline
6323)20400\\
\quad\ \ 18969
\end{array}$$

hence the square root of 10 is $3163 \div 1000$, or $3{\cdot}163$;

hence the square root of $\frac{144}{10} = \frac{12}{3{\cdot}163} = 3{\cdot}714$, etc.

But we could find this at once by adding on ciphers, which with the decimal point duly marked are valueless; thus—

$$\begin{array}{l}
\quad 14{\cdot}400000\text{ etc. }(3{\cdot}714,\text{ etc.}\\
\quad\ 9\\
\hline
67)5\,00\\
\quad\ 4\,89\\
\hline
741)1100\\
\quad\ \ 741\\
\hline
7424)35900
\end{array}$$

We insert the unit point after the 3, because we know that the square root is between 3 and 4. This shows that if we do not move the point an even number of places to the right, we may mark from the decimal point both ways, and that the decimal point in the root will correspond to that in the number itself.

14. Questions are sometimes asked like this. Find the square root of $35 \times 12 \times 28 \times 15$.

It often happens that the factors of these numbers will occur in pairs; hence the square root may be seen. Let us resolve these numbers into their factors and rearrange them. Thus—

$$\begin{aligned} &5\times 7\times 2\times 2\times 3\times 2\times 2\times 7\times 5\times 3\\ &=5^2\times 7^2\times 2^2\times 2^2\times 3^2\\ &=420^2; \end{aligned}$$

hence the square root is 420.

15. It is possible in fractions to disguise the root by inserting factors common to both numerator and denominator. Thus: find the square root of $3\frac{3}{7}\times 14\frac{2}{3}\times 2\frac{5}{8}\times 38\frac{1}{2}\times 1\frac{5}{7}\times 2$,

$$\text{reducing } \frac{24\times 44\times 21\times 77\times 12\times 2}{7\times\ 3\times\ 8\times\ 2\times\ 7}$$
$$=12\times 11\times 11\times 12, \text{ or } 12^2\times 11^2;$$

and the root of this is 132.

16. Here is a well-known geometrical fact, which is very often utilized in solving arithmetical problems. In a right-angled triangle, the sum of the squares of the numbers which measure the sides (in any unit of length) that contain the right angle is equal to the square of the number that measures the side (in the same unit of length) that is opposite the right angle.

When we treat on the measurement of plane surfaces we shall give two practical proofs of this proposition. At present we confine it to numbers.

17. How long a ladder must I have to reach a window 40 feet from the ground, if I put the foot of the ladder 30 feet from the base of the house.

This proposition tells us that (the length of the ladder)^2 = (height of window)^2 + (distance from foot of ladder to base of house)^2,

$$\begin{aligned} \text{or (ladder)}^2 &= 30^2+40^2\\ &=900+1600\\ &=2500=50^2;\\ \therefore \text{ladder} &= 50 \text{ ft.} \end{aligned}$$

18. Considering that between 1 and 100 inclusive there are only ten perfect square numbers, that is, numbers of which the square roots can be found, and that between 100 and 400

there are only ten, it is important that we should be able to find, as easily and as shortly as possible, an approximate square root. This we can do to any extent of accuracy we like by adding on pairs of ciphers—if we treat the number as an integer, and divide the resulting root by 10, that is, removing the decimal point one more place to the left.

19. If we have formed the root to any numbers of figures, say 6, we can find the next 5 (one less than 6) by dividing the remainder by twice the root already found.

The following shows this :—

Let us have to find the square root of 3·2551 correct to four decimal places. There will be five figures in the root. We will find it first in the ordinary way, and secondly, we will find the last two figures by division.

3·25510000(1·8041	·3·2551000)1·8041
1	1
28)225	28)225
224	224
3604)15100	360)1510
14416	1440
36081)68400	700
36081	360
etc. etc.	etc. etc.

If we examine these two methods, we find that the figures in the root are the same, and that in the right-hand working the last two figures (three having been already found) are found by Simple Division.

To show that this is possible, supposing the figures to be brought down are ciphers, or that the number is a perfect square, let us write the root we have found 18041 (omitting the decimal point) in this form $180 \times 100 + 41$. The square of this as shown in paragraph 8 is—

$$180^2 \times 100^2 \times 2 \times 180 \times 100 \times 41 + 41^2.$$

After we have found the 180, we have to find the number 41 from the remainder $2 \times 180 \times 100 \times 41 + 41^2$, together with a number which would appear as the remainder.

This before-mentioned remainder we divide by $2 \times 180 \times 100$, and the quotient cannot be less than 41 nor (under the supposition that the number is a perfect square, or ends in twice as many ciphers as we are finding figures by *division*) greater than 41. It is quite evident that it cannot be less than 41. If the original number is a perfect square, the quotient cannot be greater than 41, since the remainder is 41^2 (a number of four figures), which cannot contain $2 \times 180 \times 100$ (a number of five figures).

If the original number end in four ciphers, the remainder must also be less than $2 \times 180 \times 100$. If the number be very near the square of 18042, we should, of course, obtain 18042 by division. When, however, we utilize the theorem, the last figures are always ciphers, and under those circumstances we could obtain no higher number than 41.

It would be a useful service for the student to see how near to the square of 18042 it would be before the quotient would be 42 instead of 41.

20. To extract the square root of numbers expressed in other than the denary scale is a useful exercise. The same method precisely is adopted as in denary; only, of course, we divide our products by the scale number, instead of 10. Let us find the square root of the number 315432 (septenary). The first six square numbers of the scale are 1, 4, 12, 22, 34, 51.

$$
\begin{array}{rl}
 & \overset{\frown}{31}\overset{\frown}{54}\overset{\frown}{32}(453 \\
 & 22 \\ \cline{2-2}
115) & 654 \\
 & 614 \\ \cline{2-2}
1233) & 4032 \\
 & 4032 \\ \cline{2-2}
 & \cdots \\ \cline{2-2}
\end{array}
$$

315432 (septenary) will be found to be 54756 (denary), and 453 (septenary) will be found to be 234, which is the square root of 54756.

21. If any square number, as 121, or 144, or 169, is produced so that there is neither carrying in the multiplying or the adding, it will be found they express square numbers in any scale in which the figures may be used. Thus 121 is a square number in any scale whose radix is greater than 2, 144 in any whose radix is greater than 4, and so also 1 followed by an even number of ciphers is a square number in any scale whatever.

22. Since the product of the sum and difference of any two numbers is equal to the difference of their squares; if we know the difference of any two squares, we can, by resolving this difference into factors, find the numbers. Given that the difference of two squares is 21, or 3×7. The two numbers whose sum is 7 and difference 3 are 5 and 2, and $5^2 - 2^2 = 21$.

Examination and Examples.

1. Find the square of $200 + 40 + 7$, as in paragraph 8.

2. Hence find the square root of 61009.

3. Find the square root of $4\frac{5}{7} \times 4\frac{6}{11} \times 3\frac{3}{5} \times 11\frac{2}{8}$.

4. Find the square root of $5\frac{1}{11} \times 4\frac{4}{7} \times 11\frac{2}{8} \times \frac{4}{5} \times 4\frac{5}{7}$.

5. Find the square root of 804609.

6. Find the square root of ·02819041.

7. Find correct to three places of decimals the square root of 12·1.

8. Find within one thousandth the square root of 4·7.

9. Find the square root of $·09 \times 144$.

10. Find the square root of $·009 \times ·004$.

11. Find the square root of $\frac{4}{5}$.

12. Find the square root of $\frac{1}{7}$.

13. What number multiplied by itself will produce 6407522209?

14. Reduce 169 to the quaternary scale. Find its square root, and reduce it to the denary.

15. Write down the squares of the first nine numbers in the senary scale.

16. Find the perfect square numbers between 200 and 300.

17. Find, without finding the squares, the difference between the squares of 19041 and 19042.

18. Write down the differences between the first 12 consecutive squares, and show that they all differ by 2.

19. Show that any square number is exactly divisible by 4, or can be made divisible by 4 by subtracting 1 from it.

20. How many times does the square of a number contain the square of its half?

21. Remembering that the values of diamonds increase and decrease according to the square of their weight, find the weight of a diamond worth £12,278,016 if a diamond of 2 carats is worth £1.

22. Given that the difference between the squares of two numbers is 187, find the numbers.

23. If the difference between two square numbers is 4325, find them.

24. Reduce 128881 to the duodenary scale, and find its square root.

25. If the square of a number differ from the square of the next number by 487, find it.

26. Show this well-known geometrical fact with the line 58 inches long divided into 2 parts, 31 and 27—

$$31^2 + 27^2 = 2 \times (\tfrac{58}{2})^2 + 2 \times (29 - 27)^2.$$

27. Show this well-known geometrical fact with the line 117 inches produced 17 inches farther—

$$134^2 + 17^2 = 2 \times (\tfrac{117}{2})^2 + 2 \times (\tfrac{117}{2} + 17)^2.$$

28. Find the square root of quarter

$$\{134^2 + 17^2 - 2 \times 134 \times 17\}.$$

29. Given the fact that in a right-angled triangle the square on the side opposite the right angle is equal to the sum of the squares on the side containing the right angle.

If the sides containing the right angle are respectively 4 yds. and 3 ft. 6 in., find the other side.

30. A and B are apart from one another 500 yds. If by walking south and east respectively they meet, and to do this A has to walk 400 yds., how many does B walk?

31. If they arrive at the point where their paths intersect at the same time, compare their rates of walking.

32. Remembering that the areas of circles are as the squares of their diameters, if the areas of two circles are as 36590401 to 12809241, compare their diameters.

33. If the difference between the areas of two circles is 2365, compare their diameters.

34. If the difference between the prices of two diamonds is £2365, compare their weights.

35. If £1500 be worth £1591, 7s. at the end of two years, find the rate of interest.

36. Find (without squaring) the difference between the squares of 1604 and 1589.

37. If the side of a right-angled triangle which subtends the right angle be 65 feet and one of the others is 25, find the third side.

38. Find correct to three places of decimals the diagonal of a square whose side is 42.

39. Find the answer to 38 without squaring the 42.

40. Find correct to four places of decimals the diagonal of the cube whose edges are 44 ft.

41. Find the answer to 40 without squaring the 44.

42. Since the perpendicular from the vertex of an equilateral triangle bisects the base, find the length of it when the sides of the triangle are 16 feet each.

43. Find the length of this perpendicular without squaring 16 or its half.

44. A ladder 50 feet long being placed in a street, reached a window 48 feet from the ground on one side of the street, and by turning it over, without removing the foot, it reached another window 40 feet high on the other side. Find the breadth of the street.

45. A footpath goes up the side and then along the end of a rectangular field 216 yds. long and 195 broad, what distance will be saved by cutting across in the direction of the diagonal?

46. A ladder 65 feet long, placed with its foot 33 feet from a wall, reaches within 7 feet of the top. How near the wall must *the foot* of the ladder be brought that it may reach the top?

47. On one side of a square AB, 5 feet, draw a semicircle. Take a point in the semicircle D, so that AD = 3. Find DB.

48. If two triangles of exactly the same shape contain areas as 133496 : 5322249, compare the lengths of any pair of corresponding sides.

49. Find the smallest number expressed by six figures that is an exact square.

50. Find the hypotenuse expressed in the septenary of the right-angled triangle whose sides are 201 (quinary) and 125 (septenary).

CHAPTER XXII.

Measurement of Rectangular Surfaces.

1. To measure a line we must have some arbitrary unit. In England we measure long distances by miles and short distances by feet. But unless we understood mutually what we meant by a mile or a foot, the expression 3 miles or 7 feet would express nothing.

2. To measure a surface we must have as our unit some portion of surface. It would have been very inconvenient, but still, if we had chosen, we could have taken a circular portion of surface as our unit, and said that a table had an area of so many particular circles, and that a field had an area of so many particular circles, probably, though not necessarily, larger than the others. Or we might have had equilateral triangles, or hexagons, and with far less inconvenience, as our units of surface measure; but we have adopted the square, and we call it by the name of the length of its side; thus square inch, square foot, square yard, square mile.

3. Now, if you draw on a large sheet of paper a square whose sides exactly measure a yard, this is called a square yard. Mark the points where the feet divisions come, and draw lines from point to point parallel to any two sides of the square that form an angle, and you will find that you have subdivided your square yard into 9 squares, and that each of these smaller squares is exactly a foot every way, and is therefore a square foot.

To find this number 9 we multiply the 3 of the 3 feet in the one side by 3 of the 3 feet in the adjacent side, and get 9.

4. It is general to say that, though we cannot multiply pounds by pounds to get anything intelligible, it *is* possible to multiply feet by feet, and again to multiply the product by feet, and get an explicable product, but that beyond this it is not possible to go.

5. To my mind it is just as absurd to multiply feet by feet as pounds by pounds or shillings by ounces. Supposing I

want to know how many square feet there are in a rectangle which is 7 feet long and 3 feet wide.

If on the short side I measure off one foot, and draw lines through the foot divisions on the long side, it is evident I have a row containing 7 square feet; and since there are 3 feet in the short side, I can obtain three such rows, and that therefore there are 21 square feet.

Now, supposing I had a rectangular table marked out in square feet, 5 feet one way and 4 in another, and I wished to put a penny on each square foot; when I had filled one row I should have put down 5 pence, and this I should have to do 4 times; hence altogether I put down 4×5 pence, and in this case I multiply 4 pence by 5 pence as much as and no more than I multiply the 4 feet by 5 feet.

6. Practically, no doubt, it is very useful to use any little knowledge of Algebra we have, and to be able to say that 3 ft. $\times$ 5 ft. $=$ 15 ft.^{2}, and that 4 ft. $\times$ 3 ft. $\times$ 2 ft. $=$ 24 ft.^{3}, and that 14 ft.$^2 \div$ 2 ft. $=$ 7 ft., etc. etc. But we are not really multiplying or dividing the quantities, but the numbers abstracted from the quantities. Those who object so strongly to the old 'Rule of Three' method in working questions in Proportion, or discard Chain Rule as an absurdity, are recommended to examine a little more closely, than they must have done, the true nature of the operation of finding the area of a rectangular surface by multiplying the length by the breadth.

7. If we wish to find the area of a table 4 ft. long by 2 ft. 6 in. wide, let us not reduce the 4 ft., but let us change the 2 ft. 6 in. into 30 in. Now, if we draw lines through the foot points of the one side and the inch points of the other side, we shall find that we have 30 rows with 3 rectangles in each row, or altogether 90 rectangles; but these rectangles are not square, each of them being a foot long and only an inch broad; but we can easily see that in each of these rectangles there are 12 square inches; hence, if we wish to find a number of square figures exactly contained in the table, we must multiply this number 12 by the number 90, which gives us 1080 small squares an inch each way.

Supposing we reduced them both to half feet, and drew the lines through the half feet points, we should have 5 rows with

8 in each row, or 40 squares, each side being ½ a foot long. And if you draw the lines through the middle points of a square foot, you will see you obtain 4 squares of 6 in. each side.

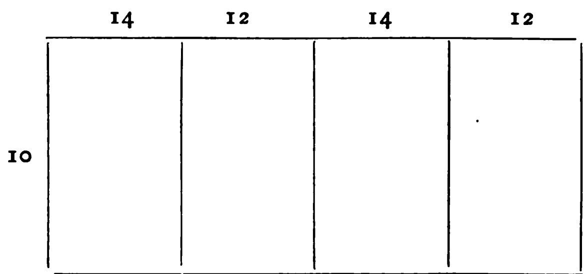
8. You can often save yourself much labour by reducing them to the highest denomination possible. As an example, supposing I want to know how much it would cost me to paint a wall 18 ft. 6 in. long, and 7 ft. 6 in. high, at 4d. a square foot.

18 ft. 6 in. = 37 half feet,
7 ft. 6 in. = 15 half feet.

The area therefore $= 37 \times 15$ square half feet, or $\dfrac{37 \times 15}{4}$ square feet, and this at 4d. a yard would cost $\dfrac{37 \times 15 \times 4}{4}$, or 37×15d.; that is, $\dfrac{37 \times 15}{12}$s., or $\dfrac{185}{4}$s., or 46s. 3d.

9. Unless we have to add or subtract some element in the course of our work, it is always as well to perform all the necessary operations by symbols until the very last. Thus—

Find the cost of papering the walls of a room 10 ft. high, 14 ft. long, 12 ft. wide, with paper at 6d. a square foot. The area we have to cover is this—



Double a piece of paper into the shape of the walls of a room, and then open it out and you will see at once what is meant.

The area is 10×52 sq. ft., and this costs $10 \times 52 \times 9$d., or £$\frac{10 \times 52 \times 6}{12 \times 20}$; that is, £13.

10. Questions of this kind are often asked: How much would it cost to paper a square room with a paper 2 ft. 2 in. wide at 3s. 9d. a dozen yards, the height being 13 ft. and the length and breadth 16 ft.?

The area of the walls is

$$13 \times 4 \times 16 \text{ sq. ft.},$$

and this is to be equal to a long piece of paper of $2\frac{1}{6}$ ft. wide. To find this length we must find what number must be multiplied by $2\frac{1}{6}$ to produce $13 \times 4 \times 16$; hence the length of the paper is $\frac{13 \times 4 \times 16 \times 6}{13}$, and this will cost, at the rate of 3s. 9d. or 45d. for 12 yards,

$$\frac{13 \times 4 \times 16 \times 6 \times 45}{13 \times 12} \text{ pence,}$$

$$\text{or £}\frac{13 \times 4 \times 16 \times 6 \times 45}{13 \times 12 \times 12 \times 20} \text{ or £6.}$$

11. Any questions amongst those appended to this chapter that require explanation will be fully dealt with in the Answers.

12. Questions of this kind worked by the duodenary scale will be given in the chapter where we deal with solids.

13. Though a square whose side is an inch is called a square inch, and those whose sides are a foot, a yard, a pole, a mile, are called square foot, square yard, square pole, and square mile respectively, there is no square furlong.

14. Since there are 8 furlongs in a mile, there would be 64 square furlongs in a square mile, or a square furlong would be the sixty-fourth of a square mile; the tenth of this is called an acre, and the fourth of an acre a rood; so that our table after poles is—

40 poles or $\frac{1}{40}$ square furlong = 1 rood.
4 roods or $\frac{1}{10}$ square furlong = 1 acre.
640 acres or 64 square furlongs = 1 square mile.

15. If we had chosen to use an equilateral triangle as our unit of surface measurement, an equilateral triangle of any size could easily be expressed in this unit. Let us adopt as our unit an equilateral triangle whose sides are 1 in. How many units are there in an equilateral triangle whose sides are 5 in.? As in rectangles, we could multiply the number which expressed the two sides about any angle, and obtain 5×5 or 25. This is known from the fact that the area of triangles vary as the square of their corresponding sides. It can be shown thus—

If in the side of an equilateral triangle whose sides are 5 in. you mark the inch divisions, and draw from point to point lines parallel to the side, you will find that you divide the triangle into 25 equilateral triangles whose sides are 1 in. each. In the lowest line of triangles between the base of the triangle and the line which joins the first division, there will be 9 triangles; 5 with their base on the base line of the big triangle, and 4 with their vertices touching the base line. In the next line there will be 7, in the next 5, in the next 3, and in the last 1, which together make up 25.

16. Again, if we chose a regular hexagon (six equal sides, each 1 in.) as our unit, and wished to measure with it another hexagon whose sides were all 4 in., we should find its area, as before, by multiplying the number expressing two adjacent sides, viz. 4×4, or 16 of the unit, or 1 inch hexagons.

This cannot be shown quite as clearly as in the case of the equilateral triangles, but if in the large hexagon, as before, you mark the inch points and draw all the lines possible through these points parallel to other sides of the figure, you will divide the figure up into 6×14 equilateral triangles, and 6 of these equilateral triangles exactly form the inch or unit hexagon. Hence the hexagon whose sides are 4 in. contain the hexagon whose sides are 1 in. 4×4, or 16 times.

The same would be true of any number of sides; but with other figures than the regular triangle, square, and hexagon it would be difficult to show it on paper by actual division, as these are the only figures which will exactly cover a flat surface, and in the case of a surface covered with hexagons *the* boundary does not consist of 6 straight lines, but of one of

many straight lines, inclined to one another in a backward and forward direction.

17. Land surveyors measure land by a chain which is subdivided into 100 links. A square chain is the tenth of an acre, and therefore contains 484 square yards, and the side of a square chain is the square root of 484, or 22 yards; a link is therefore $\frac{1}{100}$ of 22 yards, or $\frac{66}{100}$ feet, or $\frac{66 \times 12}{100}$ inches, or 7·92 inches. Hence to find the area of a rectangular field measuring 5 chains 40 links one side, and 4 chains 20 links the other, we multiply them as in common multiplication, getting 226,800 square links, or 2 acres 2 square chains and $\frac{68}{100}$ of a square chain, which, of course, can, if necessary, be reduced to square poles, yards, feet, etc.

Since $5\frac{1}{2}$ is contained in 22 four times, a square chain is 16 times a square pole. The 2·68 square chains can be at once reduced to square poles by multiplying by 16. Thus—

$$2{\cdot}68 \times 16 = 42{\cdot}08 \text{ square poles};$$

$\therefore$ the field 5 chains 40 links $\times$ 4 chains 20 links = 2 ac. 1 rd. 2·88 po.

Examination and Examples.

1. If a circle whose diameter is one inch were the unit of surface measurement, what figures could be easily measured, and how?

2. State clearly what is understood by the expression 3 ft. × 5 ft. = 15 sq. ft.

3. If I found the area of a table 2 yds. long and 3 ft. wide by multiplying the 2 by 3, what are the 6 in the product?

4. Supposing I knew that it cost 1d. to paint a surface 3 ft. by 6 in., how could I most easily get the price of painting a room 21 ft. by 5 ft. 6 in.?

5. Find the cost of carpeting a room 34 ft. 9 in. long by 26 ft. 6 in. wide, at 4s. 6d. per square yard.

6. Find the cost of papering a room 37 ft. 3 in. long, 18 ft.

4 in. wide, and 14 ft. high, with paper 2 ft. 4 in. wide, at 6d. per yard.

7. Find the quantity of carpeting required for the central portion of a room, this portion being 14 ft. 5 in. wide and 18 ft. 7 in. long.

8. Find also the cost, the carpet being only $\frac{3}{4}$ yd. wide, and 4s. 6d. a yard.

9. If between the edge of the carpet and the walls there is a distance all round of $2\frac{1}{2}$ ft., how much of the area of the floor will remain uncovered?

10. A room is 20 yds. 1 ft. 6 in. long, 15 yds. 1 ft. 6 in. wide, and 17 ft. 6 in. high. How much will it cost to paper it with paper $\frac{3}{4}$ yd. wide, at $2\frac{1}{4}$d. per yard.

11. Find the cost of carpeting a room 18 ft. 9 in. long and 17 ft. 6 in. broad, with carpet 2 ft. wide, at 4s. 9d. a yard.

12. Find the cost of papering a room of which the length is 28 ft. 6 in., the breadth 18 ft. 9 in., and the height 12 ft., with paper 1 ft. 9 in. broad, at 2s. 6d. per yard.

13. Find the cost of papering a room 25 ft. long, 18 ft. 6 in. wide, and 10 ft. high, with paper 2 ft. wide, at 3d. a yard.

14. What would be the cost of paving a hall, 50 yds. long by 50 ft. broad, with marble slabs 1 ft. long and 9 in. broad, the price of the slabs being £5 per dozen.

15. A room is a perfect cube, each wall being 15 ft. in height and 15 ft. in breadth; there is a door 6 ft. high and 4 ft. wide, and a window 7 ft. high and 3 ft. wide. Find the cost of papering the walls with paper 1 yd. wide, at 1s. 2d. per yard.

16. Find the side of a square field which contains 870,014,016 square feet.

17. A box with a lid is made of planking $1\frac{1}{2}$ in. thick. If the external dimensions be 3 ft. 6 in., 2 ft. 6 in., and 1 ft. 9 in., find how many square feet of planking are used in the construction.

18. An upholsterer covers a floor 21 ft. 8 in. by 16 ft. 6 in. with carpet 27 in. wide. Find the cost of the carpet, at 3s. $4\frac{1}{2}$d. per yard in length.

19. Find what length of paper, 14 in. wide, will cover the walls of a room 15 ft. long, 13 ft. broad, and 10 ft. high,

and find the cost if the paper is sold at three-halfpence a foot.

20. The area of a square field is 3 ac. 1 rd. 38 po. $20\frac{1}{2}$ sq. yds. Find the size of a rectangular field whose length and breadth exceed a side of the square field by 390 and by 65 yds. respectively.

21. A cistern without a lid, whose floors and walls are $1\frac{1}{2}$ in. thick, is 5 ft. 3 in. long, 3 ft. 7 in. wide, and 2 ft. $5\frac{1}{2}$ in. high in its external dimensions. Find its internal surface, and the cost of painting the same at 4d. per square foot.

22. The area of a rectangular field contains 975,744 sq. ft. One of the sides is $3\frac{1}{2}$ times as long as the other. What is the length of each side?

23. The area of a square being $122\frac{1}{2}$ acres, find the length of its side in yards.

24. The breadth of a room is half as much again as its height, its length is twice its height; it costs £5, 5s. to paint its walls at $1\frac{1}{4}$d. per square foot, what are its dimensions?

25. The length of a room is double the breadth, the cost of colouring the ceiling at $4\frac{1}{2}$d. per square yard is £2, 12s. 1d., and the cost of painting the four walls at 2s. 4d. per square yard is £35. Find its height.

26. The sides of two squares contain 77 yds. 1 ft. 9 in. and 7 yds. 2 ft. 4 in. respectively. Find the side of a square whose area is equal to the sum of the areas of the two squares.

27. If the area of a square field contain 824,464 square yards, find the length of its side.

28. A rectangular court is 120 ft. long and 90 broad, and a path of the uniform width of 10 ft. runs round it. Find the cost of covering the path with flagstones at 4s. 6d. per square yard, and the remainder of the court with turf at 6s. 6d. per 100 square feet.

29. If it costs £49, 14s. 6d. to decorate a wall-space measuring 69 ft. 4 in. by 6 ft. 9 in., what will it cost for one measuring $22\frac{1}{5}$ yds. by $3\frac{1}{3}$ yds., the style of decoration used in the second case being half as expensive again as in the first place?

30. How many planks, each $13\frac{1}{2}$ ft. long and $10\frac{1}{2}$ in. wide, will be required for the construction of a platform 54 yds. long and 21 yds. broad? What will be the cost at $5\frac{1}{2}$d. per square foot?

31. The area of a square cricket-field is 9 ac. 3 rd. 8·16 po. A running path of the uniform width of 3·9 yds. is constructed close to the boundary of the field at a cost of 4d. per square yard, and the remainder of the field is laid down in turf at a cost of 5s. 6d. per 100 square yards. Find the total cost of preparing the field.

32. The diagonal of a square courtyard is 90 ft. Find the cost of gravelling the court at the rate of 1s. for every nine yards.

33. A square field contains 3 ac. 1 rd. 13 po. $5\frac{3}{4}$ yds. Find the length of each side.

34. How long will it take a man to walk round a square field whose area is $5\frac{5}{8}$ acres, at the rate of a mile in $10\frac{2}{3}$ minutes?

35. Find the expense of lining the sides and bottom of a rectangular cistern 12 ft. 9 in. long, 8 ft. 3 in. broad, 6 ft. 6 in. deep, with lead which costs £1, 8s. per cwt., and weighs 8 lbs. to the square foot.

36. The walls of a room 21 ft. long, 15 ft. 9 in. wide, 11 ft. 8 in. high, are painted at an expense of £9, 12s. 6d. Find the additional expense of painting the ceiling at the same rate.

37. In the centre of a room 21 ft. square, there is a square of Turkey carpet, and the rest of the floor is covered with oil-cloth. The carpet and oil-cloth cost respectively 16s. 6d. and 8s. 6d. per square yard, and the whole cost of the carpet and oil-cloth is £35, 4s. 6d. Find the width of the oil-cloth border.

38. The cost of levelling a square court is £70, and if each side were 3 ft. longer the cost would be increased by £7, 3s. 6d. What is the length of the side of the square?

39. Find the cost of painting the outside of a cube whose diagonal is $4\sqrt{3}$ ft. at 6d. a square foot.

40. There are two equal areas. The length of one is 4 ft. longer than the length of the other, but the breadth 1 ft. shorter. Find their area, if the longer breadth is 10 ft.

41. Divide a field of 100 acres into two lots, so that the one lot is $2\frac{1}{3}$ times as great as the other.

42. If a piece of carpet 45 ft. long, at 10s. a yard, costs 15s. a square yard, what would it cost to cover the same with inferior carpet 2 ft. 3 in. wide at 3s. 6d. a yard?

43. Find the area of a square which is equal to a rectangle of which the sides differ by 9 ft., the answer being an integer.

44. How many acres, etc., are there in a field 3 chains 50 links long, 6 chains broad?

45. What square must I cut from a square piece of cardboard so as to reduce its area by one-fourth?

46. If the cost of laying out a square courtyard be £25, 13s., of which a square patch of grass in the middle cost 1s. a square yard, and a gravel walk 3 ft. wide round this cost 1s. a foot, and the rest was laid out with grass at the same price as the middle; find the position of the gravel walk, if you know that the expense of the path is to that of all the grass as 12 : 7.

47. If in a new country farms of 200 acres have a road frontage of 220 yards, how far back do they extend?

48. If a hexagon whose sides are all 3 ft. be the unit of measurement, what would be the measure of a hexagon whose sides were all 12 ft.?

49. How much does a farmer lose of his land if he have a path 4 ft. wide running through it $\frac{3}{4}$ mile in length?

50. If an equilateral triangle, each side 1 ft., is the unit of measurement, what is the measure of a hexagon each side 12 yds.?

CHAPTER XXIII.

Cubic Contents—Cubic and other Roots.

1. To measure space, we must have some unit which occupies some space. The most convenient unit that can be found is a cube, which is a six-sided figure, all its sides being squares.

2. A cubic inch is a figure an inch long, an inch wide, and an inch thick—that is, an inch every way. So that its top and bottom and all four sides are exactly equal.

3. If we have a solid figure of six sides, whose opposite sides are parallel to one another, or parallelopiped, as it is called, we find how many cubic units it contains by multiplying the number of linear units it is long, broad, and thick (or deep) altogether.

4. The reason of this is plain. If the parallelopiped is 4 ft. long and 5 ft. broad, and we join the feet division parallel to the sides, we divide this side into 20 squares each, measuring a foot each way. And if we take off one foot of the thickness, we shall have a layer of cubic feet, numbering 20. Supposing the solid were 3 ft. thick, we should have three such layers, or 3×20 cubic ft.; hence the solid contents of a solid—4 ft. by 5 ft. by 3 ft.—are 60 cubic ft.

5. Since 12 in. are equal to 1 foot, $12 \times 12 \times 12$ or 1728 cub. in. equal 1 cub. ft., and since 3 ft. are equal to 1 yd., $3 \times 3 \times 3$ or 27 are equal to 1 cub. yd. A fathom is $= 6$ ft.; therefore $6 \times 6 \times 6$ cub. ft. $= 1$ cub. fathom, which is all that a student need know of cubic measures.

6. It would be, as stated in Chap. II., possible to have other units to measure solids with; as, for example, we might take as our unit a tetrahedron, which is a pyramid on an equilateral triangular base, each side being equal to the base. In measuring other tetrahedrons this would be useful. Supposing soldiers' tents were shaped in this way, and we knew the quantity of air in a tetrahedron of known dimensions, we could immediately find the quantity of air contained in other tents of the same shape but of different dimensions, viz. by comparing the cubes of the sides of their bases.

7. To find the cube of a compound number, as 28. Let us, as in Chap. I., write 28 thus: 20 + 8, and proceed to multiply it by itself twice over, when we obtain (placing dots as a sign of multiplication)—

$$\begin{array}{l}
20 + 8 \\
20 + 8 \\
\hline
20^2 + 20 .8 \\
\quad + 20 .8 + 8^2 \\
\hline
20^2 + 2.20 .8 + 8^2 \\
20 + 8 \\
\hline
20^3 + 2.20^2.8 + 20 .8^2 \\
\quad + 20^2.8 + 2.20.8^2 + 8^3 \\
\hline
20^3 + 3.20^2.8 + 3.20.8^2 + 8^3 \\
\hline
\end{array}$$

8. Since the cube of 1 is 1, and of 9, 729, and the cube of 10 is 1000, and of 99, 969899,—(1) we must mark off the number whose cube root is to be found into threes; (2) we must know the cube of the first nine numbers, viz. 1, 8, 27, 64, 125, 216, 343, 512, 729. Now, let us find the cube root of 21952. (1) We know there must be two, and not more than two figures in the root; hence we mark off the last three. (2) We know the ten's figures must be two, since 21000 is between 8000 and 27000; we therefore find the cube of 20, or 8000, and subtract it from the number; and if we look at the operation of cubing above, we see that we have, after subtracting the 20^3, $3.20^2 \times$ the unit figure, and $3 \times 20 \times$ the unit figure squared and the unit figure cubed left. To put it down in the ordinary way—

$$\begin{array}{rr}
 & 21952(28 \\
 & 8 \\
3 \times 2^2 \times 12\text{, or } 1200 & \overline{} \\
3 \times 2 \times 10 \times 8\text{, or } 480 & 13952 \\
8 \times 8\text{, or } 64 & 13952 \\
\overline{1744} & \overline{\ldots} \\
\end{array}$$

the 1744×8 will, of course, give the last three terms of the product of $(20+8) \times (20+8) \times (20+8)$, as shown above.

To save a line, the 64 could be put on at the end of the 12; but if the figures in the root be small, as 2 or 3, the 4 or 9 must be written 04, 09. The 1200 is called the trial divisor, and is three times the first figure of the root squared, with two ciphers appended.

We will do a longer example in this way. Let us take 223648543. Marking, we see that the cube root is a number of three figures, and that the hundreds must be six, 223 being greater than 216 and less than 343.

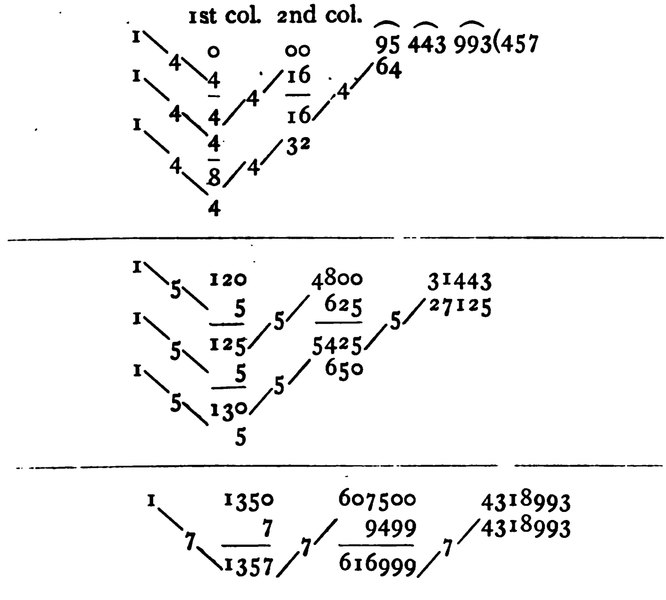
$$
\begin{array}{rr}
 & \\
 & \\
3 \times 60^2 \times 10^2 = & 1080000 \\
3 \times 60 \times 10 \times 7 = & 12600 \\
7 \times 7 = & 49 \\
\hline
 & 1092649 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
\overset{\frown}{223}\overset{\frown}{648}\overset{\frown}{543}(607 \\
216 \\
\hline
\;\;7648543 \\
\;\;7648543 \\
\hline
\;\;\;\;\cdot\;\;\cdot\;\;\cdot \\
\hline
\end{array}
$$

Here 10800 was not contained in 7648; hence we knew the next figure was a cipher. So the 6 became 60, and two more ciphers had to be added, and three more figures brought down, before we tested to see what the next figure would be.

9. Let us now find the cube root in the same form as we found the square root in Chap. I.

In square root we had only two columns, in cube we must have three, and in fifth root 5, and so on.

As before, we mark the number into sets of threes to find the figure of the highest denomination in the root. Having found this, we multiply the 1 by it, and add it on to the ciphers in column 1, and then multiply it again, and add this result on to the second column, and then again, till we get to the 216 as before. I have connected together with lines the multiplicands and their product, inserting the multiplier in the connecting lines.



10. There will be three additions in the first column, two in the second, and one subtraction in the third. Do not perform the third addition of the first column, nor the second of the first, nor the subtraction till the long line is drawn.

11. To find a fourth root. Since the fourth power is the square of the square, the easiest way is to find first the square root, and then the square root of this result; but it can be found direct as above, marking the figures into sets of four, and having four columns. Since the cube of ·1 is ·001, and that of ·01 is ·000001, etc., there must be a multiple of three figures in the decimal part of any number we wish to extract the cube root of; hence we mark off the periods of 3 from the decimal point, both to the left in the case of the units, and to the right in that of the decimals.

We will take an example, and find its cube root in both

ways, when the student can compare the different numbers that appear in the working.

To find the cube root of 17173·512.

$3 \times 2^2 \times 100 =$	1200	17173·512(258
$3 \times 2 \times 10 \times 5 =$	300	8
$5 \times 5 =$	25	
	1525	9173
		7625
$3 \times 25^2 \times 100 =$	187500	
$3 \cdot 25 \times 10 \times 8 =$	6000	1548512
$8 \times 8 =$	64	1548512
	193564	. . .

1	0	00	17173·512(258
	2	4	8
	2	4	
	2	8	
	4		
	2		
1	60	1200	9173
	5	325	7625
	65	1525	
	5	350	
	70		
	5		
1	750	187500	1548512
	8	6064	1548512
	758	193564	

We must know the fourth powers of the first nine numbers.

To find the fourth root of 43046721.

	1st col.	2nd col.	3rd col.	
1	0	00	000	43046721(81
	8	64	512	4096
	8	64	512	
	8	128	1536	
	16	192		
	8	192		
	24			
	8			
1	320	38400	2048000	2086721
	1	321	38721	2086721
	321	38721	2086721	

In the same way with five columns, and dividing one number off into sets of 5, we could find the fifth root. Such an operation might be necessary in finding the rate an amount had been calculated for five periods at compound interest. One such example will be given in the questions appended to this chapter, and worked out in the Book of Answers.

12. If we know the cubic contents of a body, and two of its dimensions, we can find the third; or if we know one of its dimensions, we can find the area contained by the other two dimensions. Students who are purposing to read higher mathematics must understand most fully this subject of measuring the solid contents of bodies.

13. In America, for the purpose of teaching the extraction of cube root, a cube is sold, which is divided into 8 pieces in this way. Three of the edges which meet at one of the angles are divided into two parts, containing say 20 and 3 units

respectively, so that the shorter divisions are all next to the angle. The cube is then sawn into 8 pieces by sawing through these divisions parallel to the three sides which contain the angle. (The student is recommended to do this for himself with a turnip, if he have not the tools to do it in wood.) The 8 pieces will be found to be 2 cubes, one of 20 units each way and the other of 3. Three of the remaining 6 pieces will be found to be square pieces 20 units each way and 3 units thick, and the other three will be 3 beams 20 units long, and the ends 3 units each way. We can find by inspection the length of each side of the big cube, and from our remainder and this length we can find the other length as shown above.

14. To students who understand the duodenary scale of notation the subject of duodecimals will present no difficulty.

15. As taught in Arithmetics, the cubic foot, square foot, and linear foot are each divided into twelve equal parts, called primes, and these primes into twelve equal parts called seconds, and these again into twelve thirds.

Hence a linear prime is a linear inch, a superficial second is a square inch, and a cubic third is a cubic inch. Since $\frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$, when we multiply primes by primes we get seconds.

To take an example. Find the area by duodecimals of a surface 3 ft. 5 in. by 4 ft. 7 in.

3 ft. 5′	$7' \times 5' = 35''$, or 2′ 11″
4 ft. 7′	$7 \times 3 + 2 = 23'$, or 1 ft. 11′
1 ft. 11′ 11″	and 4 ft. $\times$ 5′ = 20′, or 1 ft. 8′
13 ft. 8′	and 4 ft. $\times$ 3 ft. + 1 ft. = 13 sq. ft.
15 ft. 7′ 11″	

So that the answer is 15 sq. ft. $7 \times 12 + 11$, or 95 in.

16. To work one more example. Find the cubic content of a cistern 2 ft. 3 in., 4 ft. 6 in., and 1 ft. $4\frac{3}{12}$ in.

$$
\begin{array}{rrrrr}
 & & \text{2 ft.} & 3' & \\
 & & \text{4 ft.} & 6' & \\
\cline{3-4}
 & 1 & 1' & 6'' & \\
 & 9 & 0' & & \\
\cline{2-4}
 & \text{10 sq. ft.} & 1' & 6'' & \\
 & \text{1 ft.} & 4' & 3'' & \\
\hline
 & 2' & 6'' & 4''' & 6'''' \\
\text{3 c. ft.} & 4' & 6 & 0''' & \\
10 & 1 & 6 & & \\
\hline
\text{13 c. ft.} & 8' & 6'' & 4''' & 6''''
\end{array}
$$

Ans. 13 c. ft., $8 \times 144 + 6 \times 12 + 4$ c. in. $+ \frac{6}{12}$, or $\frac{1}{2}$ c. in.

EXAMINATION AND EXAMPLES.

1. If a perfect sphere were the unit of solid measurement, what figures could you measure with it, and how?

2. If I found the solid contents of a figure 2 yds. long, 2 ft. wide, and 2 ft. thick, without reducing the yards, what would be the resulting 8 got from $2 \times 2 \times 2$?

3. Supposing we chose a brick 9 in. by 5 in. by 4 in. as our unit, how could you find the number of units or bricks in a wall 90 ft. long, 10 ft. high, and 16 in. thick?

4. Supposing the unit of measurement were a tetrahedron containing 25 cubic feet, how many cubic feet would a tetrahedron contain the sides of whose base contained that of the unit ten times?

5. How many bricks 9 in. by 4 in. by 3 in. are contained in a wall 270 yds. long, 16 ft. high, and 15 in. thick?

6. The *external* length, breadth, and depth of a box are 6 ft. 2 in., 3 ft. 8 in., and 2 ft. respectively, and it is made of wood 1 in. thick. Find the quantity used.

7. A tank is 30 ft. 9 in. long, 16 ft. 7 in. wide, and 6 ft.

4 in. deep. Find how much water it will hold in cubic feet and inches.

8. A room 11 ft. high is half as long again as it is wide, and its cubical contents is $4768\frac{1}{2}$ cubic ft. Find its length and breadth.

9. Find the value of a balk of timber 39 ft. 6 in. long and 3 ft. 7 in. thick each way, at 2s. 6d. a cubic foot.

10. How many bricks, each 9 in. by $4\frac{1}{2}$ in. by 3 in., are there in a stack 36 ft. long, 9 ft. wide, and 12 ft. high?

11. How many bricks will be required for a wall 25 yds. long, 15 ft. high, and 1 ft. $10\frac{1}{2}$ in. thick, each brick being 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. deep?

12. What is the length of the edge of a cubical cistern which contains as much as a rectangular one whose edges are 154 ft. 11 in., 70 ft. 7 in., and 53 ft. 1 in.?

13. If a schoolroom is 25 ft. long and 20 ft. wide, how many children will it accommodate, allowing for each child 8 superficial feet? And if the room be 10 ft. 4 in. high, what cubical space is there for each child?

14. A cubic foot of wood weighs 20 lbs. Find the weight of 10 planks, each 30 ft. long, 1 ft. wide, and 1 in. thick.

15. If a cubic foot of marble weigh 2·716 times as much as a cubic foot of water, find the weight of a block of marble 9 ft. 6 in. long, 2 ft. 3 in. broad, and 2 ft. thick, supposing a cubic foot of water weigh 1000 ounces.

16. What is the value of a block of a certain material 5 ft. 3 in. long, 2 ft. 4 in. wide, and 1 ft. 2 in. thick, worth 4 guineas per cubic foot?

17. If the block weigh 1 cwt., what will one of the same material, 7 ft. long, 1 ft. 9 in. wide, and 10 in. thick, weigh?

18. If a cubic foot of gold may be made to cover uniformly and perfectly 432,000,000 sq. in., find the thickness of the coating of gold.

19. A decimetre is equal to 3·937 in., and a cubic decimetre of water weighs one kilogramme. If a cubic inch of water weigh 252·45 grains, express a kilogramme in pounds avoirdupois correct to within the one-thousandth part of a pound, 7000 grains being equal to one pound avoirdupois.

20. Assuming that a cubic metre contains 1000 litres, and that a metre contains 39·4 in., find the number of cubic inches in a litre.

21. Assuming that a gallon of water contains 277 cub. in., and that a cubic foot of water weighs 1000 oz., show that the popular rule, "a pint of water weighs a pound and a quarter," is nearly true.

22. A block of marble, which when squared measures 29 ft. 6 in. in length, 12 ft. 6 in. in width, and 10 ft. in depth, is valued at £407, 3s. $2\frac{3}{4}$d. At what rate is that per cubic foot?

23. What is the area of a flat roof 17 ft. 4 in. long and 13 ft. 4 in. wide? And what will be the expense of covering it with sheet lead one sixteenth of an inch thick, supposing that a cubic inch of lead weighs $6\frac{1}{2}$ oz. avoirdupois, and that a pound cost $3\frac{1}{2}$d.?

24. A block of marble at the rate of 32s. a cubic foot cost £130. It was $6\frac{1}{2}$ ft. long and $3\frac{3}{4}$ ft. wide; what would it have cost had its thickness been $2\frac{1}{4}$ ft.?

25. If a sphere be 27 times as great as another, and the radius of the former one be 12 ft., find the length of the radius of the smaller.

26. Find the cube root of 10546683057.

27. A cistern has supports that can only sustain 99 tons. It is 48 ft. long and 42 ft. broad; what depth of water can be safely poured in, if the weight of water be 1000 oz. avoirdupois every cubic foot?

28. Find by duodecimals, the number of cubic feet and inches in a tank 10 ft. 8 in long, 9 ft. 4 in. wide, and 8 ft. 9 in. deep.

29. The area of two rooms is the same, but the volume of one of them exceeds that of the other by 1080 cub. ft. If the length and height of the large room are 24 ft. and 15 ft. respectively, and the width and height of the smaller room are 18 ft. and 12 ft., what is the width of the larger and the length of the smaller?

30. Find the length of the interior edge of a cubical bin which contains 20 qrs. of wheat, if an imperial bushel fill 2218·192 cub. in.

31. Find the number of cubic chains and links in a rectangular parallelopiped whose edges are 94 chs. 50 lks., 1 ch. 5 lks., and $31\frac{1}{2}$ lks.

32. Supposing the area of a circle be $\frac{22}{7}$ times the square of the radius, find the weight of a circular disc of cast iron 7 ft. in diameter and $1\frac{1}{3}$ in. thick, it being given that a plate 1 ft. square and 1 in. thick weighs $37\frac{1}{2}$ lbs.

33. A wall 5 times as high as it is broad, and 8 times as long as it is high, contains 18,225 cub. ft. Find the breadth of the wall.

34. A decimetre wants 63 thousandths of 4 in.; how many cubic inches are there in a cubic decimetre?

35. I have to pack 1000 vols. in a box $4\frac{1}{2}$ ft. by $2\frac{1}{3}$ ft. wide and deep. If each book is 9 in. by 4 in. by $1\frac{1}{2}$ in.; how many books would have to be left out?

36. Compare the cubic contents of two vessels of the same length and breadth, but one is 10 in. deeper than the other, supposing the smaller to contain 1000 cub. in. less than the other.

37. Multiply by duodecimals 11 ft. 1′ 6″ by 6 ft. 6′ 8″ and the product by 11 ft. 8′.

38. A rough block of marble 20 ft. long, and having an average transverse section of 56 sq. yds., loses one-fourth of its volume by being shaped, and then weighs 108 tons. Find the weight in lbs. of a cubic foot of it.

39. Multiply by duodecimals 4 ft. 3′ 7″ by 2 ft. 6′ 3″ and the product by 8 ft. 9′. Reduce your answer to cubic feet, inches, etc.

40. Find the expense of lining a coffin with lead $\frac{1}{8}$ in. thick, whose interior area is 96 sq. ft., at 8d. a lb., supposing that a cubic foot of lead weighs 11,000 oz.

41. If 2 cubic in. of mercury weigh 1 lb. avoirdupois, and if 100 cubic in. of air weigh 31 grs., find how many miles in height a column of air will be that has the same weight as and stands on a base of the same area as a column of mercury 29·388 in. high.

42. Multiply in the duodenary scale 58*te* and 6*e*47.

43. A casket contains diamonds and pearls. The diamonds are worth as many times the pearls as the pearls are worth the

casket, and the value of the diamonds and the casket are £3411, 9s. and 1 guinea respectively; find the value of the pearls.

44. Of two iron balls of the same material one is 729 times as big as the other. If the diameter of the smaller be 7 ft., what is that of the greater?

45. If the compound interest on £1000 for 5 years is £610, 10s. 2·4d., at what interest is it calculated?

46. If the compound interest on £200 for 3 years is £18, 10s. 1·0896d., at what interest is it calculated?

47. If the compound interest on £250 for 3 years is £150, 8s. 0·78d., at what percentage is the interest calculated?

48. How many bricks will it take to build a wall 810 ft. long, 14 ft. high, and 18 in. thick, with bricks 9 in. by $4\frac{1}{2}$ in. by $3\frac{1}{2}$ in.?

49. If the bricks had been 1 in. shorter, and both kinds of bricks cost 7s. 6d. a 1000, find the difference in cost.

CHAPTER XXIV.

Areas of Triangles, Parallelograms, Hexagons, Octagons, and Trapezoids.

1. If through the two acute angles of a right-angled triangle I draw lines at right angles to the sides (other than the hypotenuse), I form a rectangle of which the triangle is a half. If, therefore, I know any two of the sides of a right-angled triangle, I can immediately find its area.

If I know the two short sides, the area is half the rectangle contained by these sides; thus, to find the area of a right-angled triangle whose sides about the right angle are 6 and 4 ft. respectively, we take $\frac{1}{2}$ of 6×4, or 12 sq. ft.

If I know the hypotenuse and one of the other sides, I can find the third side by what has been explained in Chap. XXII. and then proceed as in the last paragraph. Thus, to find the area of the right-angled triangle whose hypotenuse is 5 ft. and one side 4 ft.; first, we find the other side by getting the square root of $5^2 - 4^2$, or 3, and then getting $\frac{1}{2}$ of 3×4, or 6 sq. ft.

2. If we have two parallel straight lines and take any points AB in one of them and join AB with any number of points, CD, EF, etc.; in the other we form a series of triangles, CAB, DAB, EAB, FAB, etc.

Euclid proves to us that all these triangles are equal in area; hence, if we know the area of one, we know that of all. Let us therefore draw from A, AG at right angles to AB, meeting the other parallel in G. Then all these triangles are equal to GAB, and GAB is equal to half GA, AB.

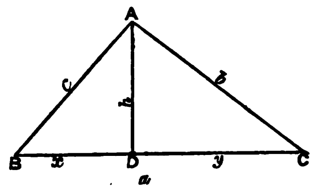
GA is called the altitude of the triangle. Hence if we know the altitude and the base of a triangle we can immediately find its area.

3. If we know an angle of a triangle measured in degrees, minutes, and seconds, and the two sides about the angle, we can take either of these two sides as the base, and published tables will enable us to find the altitude of the triangle in

terms of the other side. Supposing the angle contain 36° 50′ 20″, and one side is 6 ft. and the other 5 ft., find the area of the triangle. Let us take 6 ft. as our base. The table will tell us that the altitude is to the other side as 3 is to 5; hence the altitude is 3 ft. and the area $= \frac{1}{2} \times 3 \times 6$, or 9 sq. ft.

4. If the three sides are given the area can be found, but the rule depends on analytical calculations, and I cannot pretend to give a geometrical or ocular explanation of it.

The rule is this. Add all the sides together, halve the sum, from this half take away separately each side, and the square root of the product of these three remainders and the half of the sum of the sides will be the area of the triangle. Let ABC be any triangle of which we know the three sides, and from A draw AD perpendicular to BC. To find its area we must find the length of the altitude AD, and this, of course, we can do if we know either BD or DC. Let us call AB, BC, CA, c, a, b respectively, and AD, BD, DC, h, x, y, of which we know a, b, and c, but do not know h, x, or y; we do know, though, that $x+y=a$.



$$h^2 = c^2 - x^2 = b^2 - y^2;$$

$$\therefore\ c^2 - b^2 = x^2 - y^2 = (x+y)(x-y) = a(x-y);$$

$$\therefore\ x - y = \frac{c^2 - b^2}{a},$$

$$\text{and } x + y = a,$$

$$\text{Adding, } 2x = \frac{a^2 + c^2 - b^2}{a};$$

$$\therefore\ x = \frac{a^2 + c^2 - b^2}{2a}, \text{ and } h^2 = c^2 - x^2;$$

$$\therefore\ h^2 = c^2 - \left(\frac{a^2+c^2-b^2}{2a}\right)^2$$

$$= \frac{4a^2c^2 - a^4 - c^4 - b^4 - 2a^2c^2 + 2a^2b^2 + 2b^2c^2}{4a^2}$$

and area $= \frac{1}{2}ah$;

$$\therefore\ \text{area} = \tfrac{1}{2}\cdot a \cdot \frac{1}{a}\sqrt{\frac{4a^2c^2 - a^4 - c^4 - b^2 - 2a^2c^2 + 2a^2b^2 + 2b^2c^2}{4}}$$

$$= \sqrt{\frac{(a+b+c)\,(a+b-c)\,(b+c-a)\,(a+c-b)}{16}}$$

$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-c\right)\left(\frac{b+c+a}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)}$$

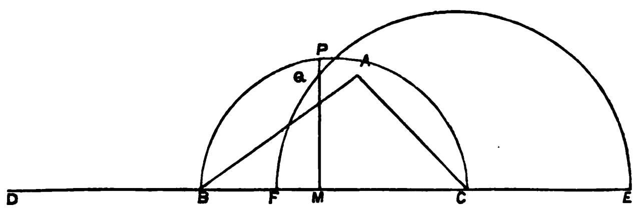
$$= \sqrt{s\,(s-c)\,(s-a)\,(s-b)}$$

where s is written for $\frac{a+b+c}{2}$.

5. The following figure will interest the student, and show what is the geometrical *meaning* of the expression, viz. :—

$$\sqrt{s,\ \overline{s-a},\ \overline{s-b},\ \overline{s-c}},$$

often abbreviated to S. Let ABC be any triangle. Produce its base both ways, and cut off BD = BA, and CE = CA. Then DE = sum of sides. Bisect DE in M. Then DM or ME = s, and BM = $\overline{s-c}$, and MC = $\overline{s-b}$. On BC describe a semi-



circle BPC, and draw MP at right angles to BC, then the

square on MP = rectangle BM, MC, or $\overline{s-b}, \overline{s-c}$, that is, MP = $\sqrt{\overline{s-b}, \overline{s-c}}$.

Again, if from DM we cut off DF = BC, then FM = $s-a$, and if on FE we describe a semicircle FQE, cutting MP in Q, then square on MQ = rectangle FM, ME, or MQ = $\sqrt{s, \overline{s-a}}$;

$$\therefore \text{MP, MQ} = \sqrt{s, \overline{s-a}, \overline{s-b}, \overline{s-c}},$$

or the rectangle PM, QM = the triangle ABC.

I am sorry to be obliged to acknowledge that I cannot show this geometrically or ocularly.

6. Since a parallelogram is double the triangle formed by any two of its adjacent sides and its diameter, we can find its area, if we know sufficient to find the area of this triangle.

E.g. find the area of the parallelogram whose two sides are 5 and 6, and whose diameter is 7. Then the area is double the triangle whose sides are 5, 6, 7.

$$5+6+7=18; \therefore s=9,$$

$$\text{and the area} = 2\sqrt{9\times 4\times 3\times 2}$$

$$= 2\times 3\times 2\sqrt{6} \text{ sq. ft.},$$

taking the square numbers from under the root.

7. To find the area of an equilateral triangle from the length of its sides, we can proceed as in 6, or we can work thus. If we call the three sides two, since the perpendicular from the vertex of triangle will bisect the base, the altitude of triangle $= \sqrt{2^2-1^2} = \sqrt{3} = 1{\cdot}73205$, that is, the altitude will contain half the base $1{\cdot}73205$ times; hence the area is

$$\tfrac{1}{2}\times(\text{base})\times(\sqrt{3} \text{ times } \tfrac{1}{2} \text{ the base}),$$

$$\text{or } \tfrac{1}{4}.\ \sqrt{3}\ \text{base}^2.$$

Let us work an example both ways. Find the area of a triangular field whose sides are each five chains.

$$\text{Areas} = \tfrac{1}{4}.25\times 1{\cdot}73205 \text{ square chains}$$
$$= 10{\cdot}82531 \text{ square chains.}$$

To work it the other way—

$$5+5+5=15\,;\ \therefore\ 8=\frac{15}{2},$$

$$\text{and } \frac{15}{2}-5=\frac{15-10}{2}=\frac{5}{2}\,;$$

$$\therefore\ \text{area}=\sqrt{\frac{15}{2}\times\frac{5}{2}\times\frac{5}{2}\times\frac{5}{2}}\ \text{sq. ft.}$$

$$=\sqrt{\frac{1875}{16}}=\frac{43{\cdot}3012,\text{ etc.}}{4}\ \text{sq. ft.}$$

$$=10{\cdot}8253,\text{ etc. sq. ft.}$$

We have worked this out in this way, but practically we should take from the root the four factors 5.5.5.5 and multiply the square root of the remaining factor 3 by 25 (the square root of $5\times5\times5\times5$).

8. Since a hexagon contains the equilateral triangle whose sides are equal to the sides of the hexagon six times, we can find the area of a hexagon by finding that of the equilateral triangle whose sides are equal to that of the hexagon, and multiplying the result by six.

9. If an octagon, whose sides are 1 unit, is divided into nine parts, by drawing through the angular points lines parallel to the sides (remembering that if the sides of a square be represented by 1, the diagonal is represented by $\sqrt{2}$; and that if the diagonal be represented by 1, the sides are represented by $\frac{1}{\sqrt{2}}$), we have 4 half squares whose sides are $\frac{1}{\sqrt{2}}$, 1 square whose sides are 1, and 4 rectangles whose sides are 1 and $\frac{1}{\sqrt{2}}$; therefore altogether the area is

$$4\cdot\tfrac{1}{2}\cdot\left(\frac{1}{\sqrt{2}}\right)^2+1+4\left(1\times\frac{1}{\sqrt{2}}\right)\ \text{square units}$$

$$=1+1+\frac{4\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$$

$$=(2+2\sqrt{2})\times\text{square on the side.}$$

Find the area of the octagon whose sides are each 8 in. ($\sqrt{2}=1\cdot4142$.)

$(2+2\sqrt{2})64=128+182\cdot2976=310\cdot2976$ sq. in.

10. To find the area of a trapezoid, that is, a four-sided figure of which two sides are parallel. Let ABCD be such a figure, and join AD; then we have two triangles of the same altitude, viz. the distance between the parallels and their bases AB, CD, and these two triangles are equal to one triangle whose base is AB and CD, and altitude the distance between the parallels. As an example, find the area of the trapezoid whose top is 6 ft., bottom 5 ft., the distance between their parallels being 3 ft., area is $\frac{1}{2}(6+5)3=16\frac{1}{2}$ sq. ft.

11. Mensuration being for the most part rather a practical than an educational science, tables are given (often without explanation) by which the areas of surfaces can be found. Such tables will not be given in this work, which is intended as an educational work, and all rules, as far as possible, will be carefully explained. In future chapters we shall show how to find the areas of certain solids, as well as of circles and other plane figures not yet treated of.

Examination and Examples.

1. Find the area of an equilateral triangle whose sides are 2 ft.

2. If we join the middle points of the triangle, what is the area of the small triangle thus formed?

3. Find the area of a triangle whose base is 8 feet and altitude 10 ft.

4. If the area of two triangles whose bases are 4 feet differ by 10 sq. ft., compare their altitudes.

5. Given that the difference of the two altitudes of two similar triangles is 3 ft., that of the area 30 sq. ft., and that of the bases 4 ft. Find the altitude of smaller triangle.

6. In a square ABCD, each side 8 ft., bisect the side and

join the middle point so as to form another square. Find the area of one of the triangles in the corners.

7. The three sides of a triangular field are found to measure 3·15, 4·17, 2·68 chains respectively. What is its area in roods?

8. If from an equilateral triangle side, 6 ft., the largest rectangle, whose base is half that of the triangle, be cut out, what is the area of what remains?

9. Find the area of a hexagonal field whose side is 2 chains.

10. Find the area of an octagon, each side 4 chains.

11. What is the length of the side of a square which exactly contains 1 acre?

12. What is the area of a triangle whose sides are 2425, 2418, and 1481 links?

13. The three sides of a right-angled triangle are 20, 21, and 29 feet. Find its area—(1) Without assuming that the triangle is right-angled, and (2) with the assumption.

14. Find the area of the square which takes me 12 minutes to walk round, at the rate of 5 miles an hour.

15. It takes me 3 minutes to walk one side of a triangular field, 4 minutes to walk the second, and 5 the third. Prove that the field is a right-angled triangle.

16. If from a square of 10 inches I take away the triangle formed by joining the extremities of the base with the middle point of the opposite side, what amount of area is left?

17. If in a hexagon of 10 feet I form a triangle by joining the extremities of the base with the middle point of opposite side, what is the area of the triangle?

18. Find the area of the parallelogram whose base is 10 ft. and altitude 4 ft.

19. If the two adjacent sides of a parallelogram be 7 ft. and 10 feet, and the smaller angle the angle of an equilateral triangle, find its area.

20. If of two triangles of the same shape the sides of the larger contain the corresponding sides of the smaller three times, show by a figure that the smaller triangle is contained exactly nine times in the larger.

21. The area of an equilateral triangle and a square are equal. Compare the length of their respective sides.

22. The area of a square and a hexagon are equal. Compare the lengths of their respective sides.

23. The area of a hexagon and an octagon are equal. Compare the lengths of their respective sides.

24. Hence, compare the lengths of the sides of an equilateral triangle, square, hexagon, and octagon, if their areas be all equal.

25. What is the area of the trapezoid whose parallel sides are 16 and 24 ft., and distance between them 6 ft.?

26. If the length of one of the other two sides is 10 ft., what is the length of the other?

27. If the length of one of the non-parallel sides were 8 ft., what is the length of the other?

28. Find the side of an equilateral triangle whose area is double the square on 2 ft.

29. Find the area of an octagon whose sides are all 1 ft. 3 in.

30. Find the sides of a triangle whose sides are as 1 : 2 : $2\frac{1}{4}$ if its area is 33·83 square in.

31. Find the difference between the area of a hexagon and an octagon whose sides are both 6 ft.

32. Give yourself the difference found in the last question, and find the length of the side common to the two figures.

33. There is a square, and on one of its sides is formed an equilateral triangle. The area of the entire figure is 22·92816 sq. in. Find the length of the side of the square.

34. Show how to find a square five times as big as another square.

35. Double down the corner of a square of 9 sq. in. so that the uncovered part may be equal to the part doubled down.

36. Cut out of the corner of a square of 16 sq. in. an isosceles right-angled triangle equal to one-third the square.

37. Draw the diagonals of a square described on a line $2\sqrt{3}$ in. Let them intersect in O, and let O be the vertex of four equilateral triangles of which the bases are on the sides, forming a Maltese cross. Find the area of the rest of the square.

38. There is a trapezoid whose parallel sides are 37 and 13 chains respectively, and distance between the parallels 25 chains. Find the area in acres, etc.

39. What is the cost of a triangular field at £4 a sq. po. if two of its sides are 5 chains each and the other 6?

40. How many *entire* triangles of an inch each side can I cut from a piece of cardboard a yard square?

41. What is the area of the cardboard left?

42. How many entire hexagons, whose sides are each 1 in., can be cut from a square yard?

43. Two pairs of opposite angles of an octagon, side 1 in., are joined, and intersect in o. Find the area of each of the eight triangles that will be formed by joining o with the angular points of the figure.

44. The exterior edge of a picture, frame and all, is 10 ft., the interior edge of the frame is 8 ft. Find the superficial area of the frame, the length being to the breadth in the ratio of 3 to 2.

45. If a triangular piece of carpet cost £5, each side being 2 yds., find the cost of a carpet of the same kind, also triangular in shape, each side being 4 yds.

46. Compare one of the six triangles which make up a hexagon with one of the eight that make up an octagon, described on equal bases, the triangles being formed as in question 43.

47. A picture hangs from a single support by a string passed through two rings in its upper edge 18 in. apart, and tied together. How much wall does the string surround if the nail is 12 in. from the picture?

48. If the string were lengthened by 20 in., how much more wall would it surround?

49. If the picture were shifted so that one of the strings from nail to picture (in. 48) were 17 ft., find the area of wall surrounded by string.

50. What would be the cost of bordering a hexagonal lawn with tiles at 9d. a foot, if the area of the lawn were $18816\sqrt{3}$ sq. ft.?

CHAPTER XXV.

Mensuration of Circles, Sectors, etc.

1. Any regular figure can be divided into as many equal triangles as there are sides, and if we know the area of one of these triangles, we can at once find the area of the figure. The point where the vertices of all these triangles meet is the centre of the figure, being equal in distance from the sides. All the triangles are isosceles. The ratio between the sides of these isosceles triangles for any particular regular figure is constant, and increases or decreases without intermission as the number of the sides increase; and this ratio can be found in tables, which are printed in books on Practical Mensuration.

2. A figure is said to be inscribed in another figure when all its angular points are on the sides of the other figure; *e.g.* draw in a circle two diameters at right angles to each, and join the four points where these diameters meet the circumference of the circle, and the square thus formed is said to be inscribed in the circle; and if through the points in the circumference lines be drawn touching the circle, the square thus formed is said to be described about the circle, and the circle is said to be inscribed in the square.

Since the described square is the square on the diagonals of the inscribed square, the former is double the latter.

3. The connection between the area of a circle and of that of a triangle, inscribed in or described about it, properly, perhaps, belongs to Trigonometry; but the former can be discovered by what has been shown with regard to the area of triangles, and one or two very patent truths in Geometry.

Let ABC be any triangle. To inscribe a circle within it. To find the centre of the circle, we have to find a point equidistant from the three sides. If we bisect any angle, every point in the bisector is equidistant from the two lines that contain the angle, therefore the point we want is in the straight line that bisects one of the angles of the triangle.

Similarly it is in the line that bisects a right angle, hence the centre must be where those two lines intersect. If we join this point with the third angle, we divide the area of the triangle into three triangles, of which the altitude is the same, viz. the radius of the inscribed circle. Hence the area of the triangle is equal to that of the single triangle, whose altitude is the radius of the circle, and whose base is the sum of the sides of the triangles; we can therefore find the radius of the circle if we know the lengths of the sides of the triangle, since $\frac{1}{2}\times$ radius $\times$ sum of sides = area of triangle.

We found the area of the triangle, whose sides were a b c, to be $\sqrt{s.\overline{s-a}, \overline{s-b}, \overline{s-c}}$, or S where $s=\frac{a+b+c}{2}$;

$$\text{therefore the radius} = \frac{S}{s}.$$

4. Let a regular figure of any number of sides be inscribed in a circle, then, from what has been said in paragraph 1, we can find its area if we know the number of sides. If the number of sides becomes very large, the altitude of the triangles is very nearly the radius of the circle, and the perimeter of the figure, or the sum of the bases of the isosceles triangles into which it can be divided, is very nearly the circumference of the circle, and, in fact, by indefinitely increasing the number of sides of the figure, we can make the difference between the perimeter of the figure and the circumference of the circle as small as we please.

The area of the figure is half the altitude of the triangles $\times$ the sum of all their bases, or $\frac{1}{2}$ the radius $\times$ the circumference; but the circumference contains the radius $2\times 3{\cdot}14159$, etc., times. Therefore the area $=\frac{1}{2}\times$ radius $\times$ radius $\times 2\times 3{\cdot}14159$ . . . or (if we call $3{\cdot}14159$ π) π radius^2.

5. To find this number $3{\cdot}14159$ . . . (the approximation of the ratio between the circumference and the diameter of a circle) requires the knowledge of higher Mathematics; but it may here be mentioned that the finding what it *exactly* is, is the well-known problem of squaring the circle, which is known to be impossible.

6. When two lines, however long, have no common unit, however small, to which they can be exactly referred, they are said to be incommensurable. The following explanation may serve to make this intelligible. Supposing the diameter of a circle were 1000 miles long, and we were to divide it into a thousand equal parts, and with this unit (viz. a mile) were to measure the circumference, we should find it was contained 3149 times with a piece less than a mile over. Now, if we were to take this piece that is over, and measure the diameter by it, we should again find a piece over; and this we might do backwards and forwards, until a microscope would be required to see the remainder, but still remainder there would be, even if we had performed the operation 1000 times.

The majority of lines that occur in the simplest and best known figures are no less commensurable than those of the diameter and circumference of a circle; *e.g.* the diameter and side of a square, the side and altitude of an equilateral triangle.

7. Since a right angle is divided into 90°, and the sum of all the angles which can be found at a point is 4 right angles, or 360°, the circumference of a circle may be said to subtend 360°.

8. A sector of a circle is a figure contained by two radii and the part of the circumference between them is called the arc of the sector or angle.

The area of sectors of the same circle are to one another as the angles their arcs subtend; therefore the area of a sector is to the area of the circle, as the angle of the sector is to 360°. *e.g.* find the area of the sector of circle, radius 3 ft., whose angle is $22\frac{1}{2}°$. Area $= \dfrac{22\frac{1}{2}}{360} \times 3\cdot14159 \times 9$ sq. ft. If we know the length of the arc which is $\dfrac{\text{angle}}{360°} \times 2 \times \text{radius}$ $\times 3\cdot14159 \ldots$, the area is immediately found by multiplying half the radius by the arc, which we leave to the student to verify.

9. To find the area of a segment, join the extremities of base with the centre to form sector, then find area of sector,

and subtract from it that of the isosceles triangle which with the segment makes the sector; thus, Find the area of segment of circle (radius 4 ft.) whose base is 6 ft., the base being 6 ft. and the radius 4 ft.; the tables will tell us that the angle of the sector is 97° 30′ very nearly, hence the area is $\frac{97\frac{1}{2}}{360} \times 3 \cdot 14159 \times 16 - \frac{1}{2} \times 6 \times \sqrt{7}$.

10. A lune is a figure bounded by the arcs of two circles, and its area is found by joining the horns of the lune (the points where the circles intersect), and finding the areas of the two segments thus formed, and subtracting that of the smaller from that of the larger. It is generally better to draw the figure with the lines, etc., in it which are given in the problem. Supposing I were asked to find the lune formed by the two circles (radii 5 ft. and 8 ft. respectively), the distance between the centres being also 5 ft., a figure would show us we should first have to find the area of a sector of the circle whose circumference is nearer the centres, and add to this area that of the quadrilateral, consisting of two equal triangles whose sides are 5, 5, 8 ft. respectively, and from this sum subtract the area of the sector of the other circle.

Knowing the three sides of the triangle, we could find the angle of the sector whose circumference is nearer the centres, and this would give us the chord of the lune, from which we can find the angle of the sector whose circumference is further from the centres.

11. If we have the data for finding the area of the two segments on the chord, of course the easier way is to find their areas, and the difference between them will be the area of the lune. If the data be as above, we must first calculate from the radii and the distance between the centres the angle of the triangles, and then get the length of the chord.

In the example in 10, the semiangle of the segment with further centre will be found in the tables to be very near to 37°; hence that of the other is 74°; and the area of triangle 5, 5, 8 is 12 sq. ft. The area of lune, therefore, is

$$\left(\frac{148}{360} \times \pi \times 5^2 + 24\right) - \frac{74}{360}\pi \times 8^2 \text{ sq. ft.}$$

12. If you draw two circles with the same centre, one larger than the other, the figure bounded by the two circumferences is a circular ring. Its area is easily found by subtracting the area of the smaller circle from that of the larger.

E.g. find the area of a circular ring 3 ft. wide whose greater circumference is $12 \times 3{\cdot}14159$ ft.

The diameter of the larger circle is 12 ft., and its radius 6 ft.; therefore its area is $36 \times 3{\cdot}14159$ sq. ft.

Again, the radius of the smaller circle is $6 - 3$ ft., and its area $= 9 \times 3{\cdot}14159$ sq. ft.; $\therefore$ the area of the ring is $(36 - 9) \times 3{\cdot}14159$ sq. ft.

13. If I draw two parallel straight lines in a circle, the figure formed by them and the circumferences (which, of course, are equal) between them is called a zone. The area is found by subtracting the area of the segment on the shorter line from that on the greater. *e.g.* find the area of the zone in a circle (radius 6 ft.) bounded by the two straight lines 3 ft. and 4 ft. We should have to have recourse to tables to find the angles of the sectors whose chords (the straight line joining the ends of the radii of a sector) are 3 ft. and 4 ft. The student is expected to draw a figure. He will see with regard to the larger segment, that the half angle of the sector is an angle of a right-angled triangle whose hypotenuse is 6 ft. (radius), and whose perpendicular (side opposite the angle) is 2 ft. Hence the table tells us that the angle (the half angle of the sector) is very nearly $19°$; therefore the area of the sector is $\frac{38}{360} . 36 \times 3{\cdot}14159$, and the area of the triangle to be subtracted from this, to find that of segment, is $\frac{1}{2} . 4 . \sqrt{6^2 - 2^2}$. The angle of the smaller sector will be found to be $28\frac{2}{3}°$; $\therefore$ the area of smaller segment is $\frac{28\frac{2}{3}}{360} \times 36 \times 3{\cdot}14159 - \frac{1}{2} . 3 \sqrt{6^2 - (\frac{3}{2})^2}$, and the area of the zone will be the difference between these two, viz.:—

$$\frac{38 - 28\frac{2}{3}}{360} \times 36 \times 3{\cdot}14159 - \tfrac{1}{2}(4 \sqrt{6^2 - 2^2} - 3 \sqrt{6^2 - (\tfrac{3}{2})^2}).$$

EXAMINATION AND EXAMPLES.

1. What do you understand by incommensurable quantities?

2. Are the sides of a right-angled triangle, of which 3 and 4 are the lengths of the sides containing the right angle, commensurable or not?

3. State clearly what you understand by the expression 'squaring the circle.'

4. How could you ocularly show a junior class in a school that the value of π is between 3 and 4?

5. The perimeter of a regular figure of twenty sides is 10 ft., and one of the long sides of one of the isosceles triangles into which it can be divided is $\frac{5}{3}$ ft. Find the area of the figure.

6. Show that to multiply the half radius by the length of the arc is the same as to find the fraction of the area of the circle that the angle of the sector is of 360°.

7. What is the radius of a circle whose circumference is 100 yds.?

8. How many degrees are there in the angle at the centre of a circle (radius 10 ft.) which subtends an arc of 8 ft.?

9. Find the chord of the arc mentioned in 8.

10. Let ABC be an arc of a circle (radius 8 ft.) 4 ft. long, B its middle point. From B drop BD perpendicular to AC, and find its length.

11. Find the area of the sector of circle (radius 6 ft.) whose arc is 3 ft.

12. Find the angle at the centre of a circle (radius 1 ft.) which subtends an arc of 1 ft.

13. Find the area of the sector of circle (radius 100 miles) whose angle is 45°.

14. Find the area of segment of circle (radius 10 yds.) whose base is 8 ft. The angle of the sector will be 15° 20′.

15. Find the area of sector of circle (radius 4 ft.) whose arc is 2 ft.

16. Find the area of the circular ring, 3 ft wide, of which *the larg*er circumference is 100 ft.

17. Find the area of the circular ring, 2 ft. wide, of which the inside circumference is $(4 \times \pi)$ yds.

18. Compare the areas of the two adjacent rings, each 2 ft. wide, the inner circumference of the smaller ring being 100 yds.

19. If the areas of two circular rings be as 10 : 3, find area of smaller ring, the width of either ring being 1 ft., and the radius of the larger circumference of larger ring being 10 ft.

20. Draw a lune, with arcs of two circles, radii 8 ft. and 6 ft., with 6 ft. between the centres, and find the length of the straight line which joins the horns.

21. Find the area of lune in 20.

22. What are the limits of the lengths of the two radii, and of the distance between the centres?

23. What is the diameter of a circle whose area is 113·09724 sq. ft.?

24. Compare the area of a circle, radius 4 ft., and that of a hexagon, each side 4 ft.

25. Find the area of what is left if a hexagon (each side 6 ft.) is cut from a circle whose diameter is 12 ft.

26. Compare the areas of the inscribed and circumscribed square of a circle whose radius is 10 in.

27. On the four sides of a square of 6 ft., semicircles are drawn inside the square by forming four double segments in the form of a star. Find the area of the star.

28. A circular playground, consisting of grass, 100 yds. in diameter, has a path on the outside, the area of which is $\frac{1}{2}$ an acre, find the width of the path.

29. The perimeter of a quadrant (a sector whose angle is $90°$) is 100 ft., find the radius of the circle.

30. If of a bicycle the larger wheel revolves as often in 10 seconds as its rate in miles per hour, find the diameter of wheel.

31. Two boys run round a circular course in opposite directions, their rates being as 3 : 5. They meet for the first time at the spot whence they started, after the faster runner has run 10 miles. Find the diameter of the course.

32. What is the area of a sector of circle whose angle is $45°$ and whose arc is 5 ft.?

33. Compare the perimeters of a square and a circle each containing 3 acres.

34. Find the expense at 10s. a rod of fencing a rectangular area of 3 acres, the ratio between length and breadth being 5 : 3.

35. Find the area of an equilateral triangle inscribed in a circle, radius 10 ft.

36. Find the area of the circle inscribed in the triangle whose sides are 4, 5, 7 ft. respectively.

37. Each side of a rhombus and its shorter diagonal is 4 ft., find its area.

38. Compare the areas of the triangle whose sides are 13, 14, 15 ft., and the circle with the same perimeter.

39. If the perimeter of a semicircular flower-bed be 100 ft., how many *entire* square feet can be measured in it?

40. What is the diameter of a bicycle wheel which makes 300 revolutions in a mile?

41. The diameter of a circle is 12 ft.; find the area of a square inscribed in it.

42. Find the area of the largest circle that can be inscribed in a semicircle, radius 10 ft.

43. Find the area of the square inscribed in the semicircle whose radius is 8 ft.

44. Find the area of the square inscribed in a quadrant (see 29) whose radius is 6 ft. (two sides of the square being coincident with the radii).

45. The area of an equilateral triangle is 100 sq. yds.; find its perimeter.

46. The area of a county containing 100,000 acres covers 5 sq. in. on a map. What is the scale by which the map is drawn?

47. The difference between the square inscribed in and described about a circle is 400 sq. yds.; find the radius of circle.

48. Find the area of segment of circle, radius 10 ft., the angle of sector being 60°.

49. Find the area of a rectangle inscribed in a circle (radius 100 yds.) whose length is double its breadth.

50. A quadrant and a semicircle have the same perimeter *(100 ft.)*; compare their areas.

CHAPTER XXVI.

Solid Contents of a Prism, a Cylinder, a Pyramid, a Cone, and a Sphere, with the Areas of their Surfaces.

1. A prism is a solid, whose ends are any two parellel plane figures, equal, similar, and similarly placed, and whose sides are determined by straight lines joining the corresponding points in the perimeter of the two ends.

2. Since all the straight lines joining the corresponding points are equal, if we know the length of one of them and the area of one of the ends, the solid content of the prism, if 'right' (see 4), will be the product of these two numbers.

3. If the ends of the prism be circles, the solid becomes a cylinder; hence if we call the straight lines which join the points of the two circles the height, its solid content again is the area of end × the height.

4. The straight line which joins the centres of the ends of a prism or cylinder is called its axis. If the axis be perpendicular to the ends, it is called a right prism or cylinder, and if not perpendicular, oblique.

5. Let ABCD be a square, and let E be its centre. From E draw EF perpendicular to plane ABCD. If we join ABCD with F, any point in EF, and consider these lines the boundaries of triangle, we form a solid figure called a regular right pyramid.

6. The base can be of any shape, and the line EF drawn from any point and at any angle to the base; but such pyramids are oblique and irregular, and beyond the scope of this manual.

7. If the base be a circle, and EF be drawn from its centre at right angles to the base, the pyramid becomes a cone.

8. If EF be drawn obliquely to the base, the cone is said to be oblique.

9. A sphere is a solid, every point of whose surface is equidistant from a point within the solid, called its centre.

10. To find the solid contents of a pyramid, we must know

that of the prism whose parallel sides are the same as the base of the pyramid, and whose length is the same as the height of the pyramid. Euclid xii. 7 proves to us that the solid contents of this prism are three times that of the pyramid.

11. Similarly the content of a cone is one-third of that of the cylinder of the same base and height.

12. The area of a sphere is the circumference × the diameter; but the reason for this depends on theorems which cannot be discussed without a knowledge of the Integral Calculus.

13. The solid content of a sphere may be considered as the sum of an infinite number of equal pyramids whose altitude is the radius of the sphere, and the sum of whose base is the area of the sphere, and this will be found to be $\frac{1}{3}$ of rad. × diam. × circumference $= \frac{4}{3}$ rad.$^3 \times \pi$, or $\frac{(\text{diam.})^3 \times \pi}{6}$.

14. The area of the sides of a prism is the sum of the rectangles whose sides are the sides of the parallel ends and the length of the prism. *e.g.* the area of the triangular prism whose base is a triangle, with sides 3 ft. 4ft. and 5 ft., and length is 8 ft., is $3 \times 8 + 4 \times 8 + 5 \times 8$, or 12×8 sq. ft.

15. The area of the circular surface of a cylinder is the circumference of the base × the height. If a piece of paper is wrapped round a cylinder, this will become evident at once.

16. The area of a cone is that of a sector of a circle whose circumference is that of the base and whose radii are the distance from the vertex to the outside of the base, which is evidently the hypotenuse of a right-angled triangle of which the other sides are the radius of the base and the height of the cone. *e.g.* find the area of the cone, the radius of the base being 9 ft. and the height 12 ft. The length of the distance from the vertex to the circumference of base is $\sqrt{144+81}$, or 15 feet, and the circumference of base is 18π;

$\therefore$ the area $= \frac{1}{2}.15 \times 18\pi$, or 135π sq. ft.

17. The solid contents of any regular solid is the sum of the pyramids whose bases are the sides of the solid and

whose heights are the distance from the sides to the centre of the solid.

18. There are only five regular solids, viz. :—

(1) The tetrahedron, bounded by 4 equilateral triangles.
(2) The hexahedron or cube, bounded by 6 squares.
(3) The octohedron, bounded by 8 equilateral triangles.
(4) The dodecahedron, bounded by 12 regular pentagons.
(5) The icosahedron, bounded by 20 equilateral triangles.

Examination and Examples.

1. What are the solid contents of a pyramid whose base contains 5 sq. ft. and whose height is 4 ft.?

2. What is the length of the straight line from the vertex to the corner of a square pyramid, altitude 4 ft., side of base 6 ft.?

3. If the length of the straight line from the vertex of a regular pyramid on a square base to one of the angles of the base be 10 ft. and the height of the pyramid 6 ft., find its solid content.

4. Find the area of the five sides in 3.

5. Find the circular area of the cone whose height is 6 ft. and the radius of whose base is 2 ft.

6. Find the solid content of the cone in 5.

7. Find the area of the sphere whose radius is 8 ft.

8. Find the solid content of the sphere in 7.

9. Find the solid content of the cylinder whose base is a circle of radius 8 ft. and height 16 ft.

10. Compare this with the answer of 7, and obtain a formula for finding the area of a sphere from that of the surrounding cylinder.

11. If the area of a sphere be 1 acre, find its diameter correct to inches.

12. Find the solid content of the sphere in 11.

13. Find the solid content of the regular pyramid on a hexagonal base, each of whose sides is 3 ft. and height 6 ft.

14. Find the solid content of the triangular pyramid, each

of the sides of the base being 4 ft. The distance of the vertex of the faces from their bases being $\frac{5}{4}$ of the altitude of the pyramid.

15. What is the solid content of the earth, whose circumference is 24,000 miles? Express your answer in cubic miles.

16. What is the weight of a sphere of water whose diameter is 10 ft., if a cubic foot of water weigh 1000 oz. avoirdupois?

17. If ivory is 1·820 times as heavy as water, find the weight of a billiard ball $1\frac{3}{4}$ in. in diameter.

18. If I cut the largest ball possible from a cube of ivory, each side 2 in., what is the weight of the parts cut away?

19. Find the solid content of the triangular prism, the sides of the triangle being 10, 10, 12 in. and the length 4 in.

20. Find the area of all the sides of the square pyramid, the edges of the square being 5 ft. and the length of the sides of the triangles 8 ft.

21. Give yourself the answer of 20, and the sides of the square. Find the length of the sides of triangle.

22. An oblique prism whose parallel ends are squares (side 6 ft.) is inclined at 60° to one of the other sides; find its solid content if the length of the prism be 8 ft.

23. If the weight of iron be 9 times that of water (see 16), what is the weight of 10 miles of cable whose section is 2 square in.?

24. What is the weight of 2 ft. of water lying at the bottom of a cylinder the radius of whose base is 8 ft.?

25. Mercury weighs 13 times what water does. Find the weight of a column of mercury whose base is 1 square in. and height 29 in.

26. If the column of mercury in the last question be the weight of the air on every square inch of a body, what would be the pressure on a circular glass globe (diameter 5 in.) from which the air had been extracted?

27. In a lifting pump, if the height of the tube be 50 ft. and the area of its section 2 in., what weight of water has to be lifted at each stroke?

28. Find the solid content of a circular stone pillar, the top being hemispherical, its extreme height being 8 ft. and *radius of* base 1 ft.

29. A cone revolves on a circular plane (radius 5 ft.) with its vertex in the centre. If in making the entire circuit of the circle it revolves five times, find the height of the cone.

30. A cylinder with two hemispherical ends is 10 ft. in its extreme length. If its circumference is 2 ft., find its solid content.

31. Find the area of the solid in 30.

32. How many times would an inking cylinder 4 ft. long have to revolve so as to come in contact with every point of a plane 4 ft. by 10 ft., if its radius is 2 in.?

33. Compare the areas of a cube and a sphere whose solid contents are both 1 cub. ft.

34. The solid content of an icosahedron is found by multiplying the cube of one of its edges by 2·1816. Find that of one whose edge is 3 in.

35. Find the length of the edge of an icosahedron whose solid content is 1 cub. ft.

36. Two rules are often given for finding the solid content of a sphere, viz. (1) Multiply the cube of the diameter by $\frac{\pi}{6}$. (2) Multiply one-third of the surface by the radius. Prove that they must give similar results.

37. If from a cone (radius 4 ft.) of height 10 ft. the upper 4 ft. are cut away, what is the solid content of the frustum remaining?

38. What would it cost to cover the frustum in 37 with silver $\frac{1}{4}$ in. thick, if silver is $10\frac{1}{2}$ times as heavy as water, and is worth 5s. an oz.?

39. Find the solid content of a frustum of a cone 4 ft. high after 1 ft. has been cut off from the top, the circumference of the base being 6 ft.

40. The angle which the axis makes with the slant height of a cone is called the semivertical angle. Prove that cones with the same vertical angle are to one another as the cubes of the radii of their bases.

41. A cone 10 ft. high, radius of base 3 ft., is covered with iron 1 in. in thickness on its curved area, so as to form a larger cone. Find the new height.

42. Find the solid content of the iron put on to the cone in 41.

43. A pyramid on a square base (side 100 yds.) whose height is equal to the diagonal of the base is stopped in its building when 20 ft. from the top. Find the quantity of stone in cub. po. used in the part finished.

44. If the part finished be covered with marble 1 in. thick, how much marble would be required?

45. If gold weigh 20 times as heavy as water, and 1 lb. be made into 50 coins, how many coins could be made with a cylinder of gold 10 ft. high, radius of base 1 ft.?

46. How many exact spheres 1 in. in diameter could be cast from a cubic foot of metal?

47. To cover a sphere of silver 2 in. in diameter with a coating of gold cost 10 coins (see 45), how thick is the coating?

48. Tin is 7·3 times and gold 19·4 times as heavy as water. Find the area of a solid tin sphere which weighs the same as a sphere of gold, radius 2 in.

49. If iron weigh 7·8 times as heavy as water, and it is known that a gold ring is enclosed in a cylinder of iron, how could you find out without breaking the iron cylinder how much gold there was within it?

50. A cylinder of metal (diameter of base 1 in.) is surrounded with another metal $\frac{1}{2}$ an inch thick. Suppose 2 in. of this cylinder were rolled out into a wire a mile long, and the outer metal were removed by chemical action, what would be the diameter of the wire left?

CHAPTER XXVII.[1]

Exchange—Finding Specific Gravity and Problems in Physics.

1. Exchange is the equating the sums of money of different countries.

2. The rate of exchange between two countries is the connection between the standard units of money of these countries, and this connection varies from day to day according to circumstances.

3. If I purchase goods from another country, I can pay for them either by sending the gold or silver which they are worth, or I can give what is called a bill of exchange.

4. If I send coined gold or silver, I must know what the value of my gold or silver is considered to be in the country to which I send it, and this depends on the fineness or, in other words, purity of the metal of which the coins of my country are made.

5. Great care being taken in the coining of gold and silver coins, the relative intrinsic value of the coins of two countries does not vary, and the fixed ratio between them is called the 'par of exchange.'

6. In England coined gold is of the same value as bar gold, but coined silver is worth ·1 more than bar silver; coined silver is therefore never exported; as, of course, it would not be received except for its true value, and the sender would thereby lose 10 per cent.

7. A bill of exchange is an order given to the sender of the goods, which he can cash in his own country.

8. If the exports from a country are greater than the imports, bullion, of course, must be sent by some of the merchants of the country receiving the most goods, and to avoid this

[1] Before writing this chapter, I consulted freely Mr. Brooksmith's admirable Manual, and have assumed his data, for which I cordially thank him.

expense they will pay rather more for the bills which they send to their creditors than they are actually worth. On the other hand, the merchants on the other side will be able to get bills of exchange for as much less than their actual value.

9. This difference of value is called the course of exchange, and it is said to be at a premium or a discount according as it is above or below the par of exchange.

10. Find the par of exchange between the U.S. gold eagle, which contains 232 grains of pure gold, and the sovereign, of which 1869 contain 40 lbs. of gold $\frac{11}{12}$ fine. This may be made a simple question of reduction, of which our table is as follows:—

FACTS.	TABLE.
U. S. eagle contains 232 grs. fine gold.	1 eagle = 232 grs.
5760 grs. = 1 lb.	1 gr. = $\frac{1}{5760}$ lbs.
11 lbs. fine gold = 12 lbs. of B. gold.	1 lb. fine gold = $\frac{12}{11}$ lbs. British gold.
40 lbs. = 1869 sovereigns.	1 lb. = $\frac{1869}{40}$ sov.

The eagle therefore $= 232 \times \frac{1}{5760} \times \frac{12}{11} \times \frac{1869}{40}$ sov.
$= 2{\cdot}053$ sovs.,

and 1 dollar (gold) = £$\cdot 2053 = 4{\cdot}106$ shil.

11. The nominal value of the dollar is 4s. 6d., and since $\frac{4s.\ 6d.}{4s.\ 2d.} = \frac{108}{100}$, when the dollar is really worth 4s. 2d., the rate of exchange on New York is said to be 8 per cent. premium.

12. Find the arbitrated rate of exchange between Vienna and London when the exchange between Paris and Vienna is 198 francs for 100 florins, between Paris and Berlin 363 francs for 100 thalers, and between Berlin and London 6 thalers 27 silber groschen (30 s. g. = 1 thaler) for £1.

Our table is easily formed thus:—

1 English pound $= 6\frac{9}{10}$ Prussian thalers.
1 Prussian thaler $= \frac{363}{100}$ French francs.
1 French franc $= \frac{100}{198}$ Austrian florins;

$$\therefore \text{£}1 = \frac{69}{10} \times \frac{363}{100} \times \frac{100}{198} \text{ Austrian florins}$$
$$= 12 \text{ florins, } 65 \text{ cents.}$$

13. Calculate the par of exchange between the English shilling and the American silver dollar when British standard silver is 5s. $0\frac{5}{8}$d. per oz., if 1 lb. of silver $\frac{37}{40}$ fine is coined into 66 shillings, and a dollar of silver $\frac{9}{10}$ fine weigh $412\frac{1}{2}$ grains.

Since the intrinsic value of an English shilling is $\frac{60\frac{5}{8} \times 12}{66}$ pence, or $\frac{485}{8 \times 66}$ shillings,

an English shilling $= \frac{8 \times 66}{485}$ twentieths of a sov.

1 twentieth of a sov. $= \frac{1}{66}$ lb. Eng. st. silver.
1 lb. Eng. st. silver $= \frac{37}{40}$ lbs. fine silver.
1 lb. fine silver $= \frac{10}{9}$ lb. Am. st. silver.
1 lb. Am. st. silver $= 5760$ grains.

1 gr. $= \frac{1}{412\frac{1}{2}}$, or $\frac{2}{825}$ dollar;

$$\therefore \quad 20 \text{ shil.} = \frac{20 \times 8 \times 66 \times 37 \times 10 \times 5760 \times 2}{485 \times 66 \times 40 \times 9 \times 825} \text{ dollars}$$
$$= 4{\cdot}7345 \text{ dollars.}$$

Similarly, to compare English and French silver when English standard silver is worth 5s. $1\frac{3}{8}$d. an oz.; supposing that a kilogramme (15432 grains) is coined into 200 francs, French silver being $\frac{9}{10}$ fine. In this case the intrinsic value of a shilling is $\frac{8 \times 66}{491}$ twentieths of a sovereign;

$$\therefore \quad 20 \text{ shil.} = \frac{20 \times 8 \times 66 \times 37 \times 10 \times 5760 \times 200}{491 \times 66 \times 40 \times 9 \times 15432} \text{ francs}$$
$$= 25{\cdot}001 \text{ francs.}$$

14. I could, as has so often been done previously in this manual, give myself the answer of either of these questions and all the other elements except one, and find it. It will be noticed that the number of coins made from the lb. of British

silver does not enter into the calculation, as it appears twice, and they cancel one another.

15. How many ounces of bar (uncoined) gold, $21\frac{1}{2}$ carats (24 carats = pure gold) fine, will be required to be sent to pay a debt of 13330 francs, if 1 kilogramme (15432 grains) of gold $\frac{9}{10}$ fine = 3100 francs?

This is simply a question in Compound Proportion, of which the statement is as follows:—

$$\left.\begin{array}{r} \dfrac{21\frac{1}{2}}{24} : \dfrac{9}{10} \\ 3100 : 13330 \end{array}\right\} = 15432 \text{ grs.} : \text{grains required.}$$

16. Everybody knows that a volume of one material does not often weigh the same as the same volume of another material. The particular weight of a body compared with the weight of the same volume of some standard material is called its specific gravity. In a table of specific gravities the weights are compared with that of water. If, therefore, we know the weight of some volume of water, and the specific gravity of a body, we can tell the weight of any volume of that body.

One cubic foot of water weighs 1000 oz. The specific gravity of iron is 7·8. If, therefore, we are asked what is the weight of 10 cub. ft. of iron, the answer is $10 \times 7{\cdot}8 \times 1000$ oz. The question may be asked and stated as one of Compound Proportion thus: If 1 cub. ft. of water weigh 1000 oz., what is the weight of 10 cub. ft. of iron, which weigh 7·4 times as heavy as water? The statement is—

$$\left.\begin{array}{r} 1 \text{ cub. ft.} : 10 \text{ cub. ft.} \\ 1 : 7{\cdot}4 \end{array}\right\} = 1000 \text{ oz.} : \text{oz. required.}$$

Ans. $10 \times 7{\cdot}4 \times 1000$ oz., as before.

17. Though it is true that the weight of a body varies as its volume × its specific gravity, and with certain units we may venture to say that weight = volume × specific gravity, yet it is dangerous to write this down, as the units have to be so adjusted that the constant multiplier, to change the product of the volume and the specific gravity into weight, must be unity. Let us work another example. Find the specific gravity of a body, if 4 cub. ft. weigh 6000 oz.

Let V_x, W_x, S_x be the volume weight and specific gravity of the body, and let V_w, W_w, S_w be the volume, etc., of the standard substance water. Then—

$$V_wS_w : V_xS_x = W_w : W_x$$

Now we want to find the constant multiplier which will enable us to find W_x from V_xS_x.

By the principles of proportion—

$$W_x = V_xS_x \times \frac{W_w}{V_wS_w}.$$

Now for $\frac{W_w}{V_wS_w}$ to be 1. Our unit of weight must be 6000 oz., our unit of volume $\frac{1}{4}$ cub. ft., and our substance, the standard substance whose specific gravity is 1. If asked for our linear unit, it would be $\sqrt[3]{\frac{1}{4}}$, or $\frac{1}{2}\sqrt[3]{2}$ ft.

18. Since weight varies as the product of the volume and the specific gravity, the volume varies directly as the weight and indirectly as the specific gravity, and the specific gravity varies directly as the weight and indirectly as the volume. To compare the weights, volumes, or specific gravities of bodies is not hard. *e.g.* compare the specific gravity of three bodies whose weights are as $4 : 5 : 6$, and whose volumes are as $6 : 5 : 4$.

$$\begin{aligned} S_1 : S_2 : S_3 &= \frac{W_1}{V_1} : \frac{W_2}{V_2} : \frac{W_3}{V_3} \\ &= \frac{4}{6} : \frac{5}{5} : \frac{6}{4} \\ &= 4 : 6 : 9. \end{aligned}$$

19. The finding the specific gravity of bodies is a simple arithmetical operation, depending on a common-sense theory of hydrostatics. If I weigh a piece of iron first in air and then in water, I find that it weighs less in the water than it did in the air. What has become of the lost weight? The water is partially supporting the iron; but before the iron was in the water the place now occupied by the iron was of course filled with water, and the water surrounding it was supporting

this water; hence the weight lost by weighing it in water must be the weight of exactly the same volume of water, which is the very thing we want to find. *e.g.* the weight of a cubic foot of oak is 75·6 lbs., but when immersed in water it only weighs 12·6 lbs., what is its specific gravity? Since it loses 63 lbs., the weight of a cubic foot of water is 63 lbs.

$$63 : 75{\cdot}6 = S_w : S_o$$

$$\text{or specific gravity of oak} = \frac{75{\cdot}6}{63} = 1{\cdot}2,$$

or $\frac{1}{5}$ heavier than water.

It is not necessary that we should know the cubic content of the wood.

20. If the specific gravity of a body be less than 1 (that of water), the body will of course float. We should then have to sink it by means of a heavy body, whose weight in and out of water must be known.

E.g. a body weighs in air 1 lb.; the sinker weighs in air 10 lbs., and when weighed in water 8 lbs., but when the light body and the sinker are immersed together they weigh only 6 lbs. What is the specific gravity of the body? Here (not to reduce it to a formula, but to a matter of common sense) the upward pressure of the water not only supports the lb. of the light body, but 2 lbs. (8 – 6) of the heavy body; hence the water is three times as heavy as the body, and the latter's specific gravity is said to be $\frac{1}{3}$ or $\cdot\dot{3}$.

21. We can of course compare fluids by weighing the same body in each, and in air, and comparing the two differences.

22. If in a floating body we can measure the volume immersed, we can of course at once compare the specific gravity of the fluid and the solid.

23. Iron floats in mercury with $\frac{39}{68}$ of its volume immersed.

$$S_i : S_m = 39 : 68.$$

If we knew either of them with respect to the standard substance (water), we could immediately express the other. *e.g.* s. m. of mercury = 13·6;

$$\therefore \text{ specific gravity of iron} = \tfrac{39}{68} \text{ of } 13{\cdot}6 = 7{\cdot}8.$$

EXAMINATION AND EXAMPLES.

1. Explain the difference between par of exchange and course of exchange.

2. Tell clearly the nature of the operation when a London merchant purchases goods from a Paris one, paying for them by means of a bill of exchange.

3. Why would it be very unwise to export coined silver to pay for goods received from another country?

4. Compare the expressions—gold 22 carat fine and $\frac{37}{47}$ fine.

5. What is the course of exchange if £688, 14s. 8d. is worth 17511 fr. $42\frac{1}{2}$ c.?

6. How many £ must be given for a bill of exchange on New York for \$11315·$34\frac{4}{5}$, exchange being $8\frac{1}{2}$ per cent. in favour of England, the par value of the dollar being 4s. 6d.?

7. If £10 = 121 florins 16 kreuzers (60 kreuzers = 1 florin), exchange £1000 into florins and kreuzers.

8. If on leaving England a man exchanges £100 into French francs 25 to the £, and on arriving at Munich he exchanges the francs into German marks, 100 francs = 97 marks, how many would he receive?

9. If the German mark is worth 1s., how much has he gained by the double transaction?

10. How many oz. of bar gold, 22 carat fine, will he require to pay a bill of \$1468 when exchange is at 8 per cent. in favour of England? See par. 10.

11. What is the value of British silver if the par of exchange between the shilling and the dollar is 20s. = \$4·7345, if 1 dollar weigh $412\frac{1}{2}$ grains, and is $\frac{9}{10}$ fine, and 1 lb. troy standard silver, $\frac{37}{40}$ fine, is coined into 66 shillings?

12. Give yourself the answer of 11, and all the data except the fineness of American silver, and find it.

13. If 28·96d. = 1 Russian rouble, or 3 francs 9 cents, compare the French franc and the English £.

14. Give yourself the answer of 13, and find the connection between the French franc and the Russian rouble.

15. A merchant in New York wishes to remit to London

\$5110, for what sum in English money must he draw his bill, when bills *on* London are at a premium of $9\frac{1}{2}$ per cent.?

16. The Guernsey pound contains 18 oz. avoirdupois, and the Guernsey shilling 13 English pence. If a Guernsey pound of butter cost 1s. 6d. Guernsey money, what will be the price in English money of $2\frac{1}{8}$ lbs. avoirdupois?

17. If we adopted a decimal coinage with a florin as our unit, which we divided into 100 cents, what would be the value of all our present coins?

18. If we adopted £1 as our unit, and reduced our farthings to the one thousandth of a pound, and our penny to the $\frac{1}{250}$ of a pound, what would the silver coins be worth in cents and milles?

19. Pay a bill of £75, 4s. $2\frac{1}{2}$d. according to method explained in 18.

20. What would be the exact value of a bill which was paid (according to 18) with £10, ten shillings, ten pence, and five farthings?

21. When you say that the specific gravity of a body is 7·5, what do you mean?

22. If a body weigh 8 lbs. in air and only 7 lbs. in water, what is its specific gravity?

23. If two bodies weigh exactly 1 lb. in air, but the one weighs 14 oz. and the other 12 in some fluid, compare their specific gravities.

24. Three bodies weigh 5 lbs., 6 lbs., 7 lbs. respectively, and contain 7, 8, 9 cubic inches; compare their specific gravities.

25. The specific gravity of four bodies are as 4 : 3 : 5 : 6, and their volumes as 5 : 2 : 3 : 7; compare their weights.

26. Under what circumstances will solid iron float?

27. Bodies decrease in volume with their temperature, but water ceases to decrease in volume when near to freezing, and as it approaches freezing increases in bulk. What would happen if this were not so?

28. A piece of cork weighs 1 lb. in water, a sinker weighs 10 lbs. in air, but only 6 lbs. in water, but when attached to the cork it weighs 9 lbs. in water, what is the specific gravity of the cork?

29. A piece of wood is entirely immersed, floating with its

upper surface just touching the surface of the water; what is its specific gravity?

30. What is the weight of a cubic inch of iron, specific gravity 7·8?

31. Supposing on weighing it I found it weighed ½ an oz. more than the answer of 30, what could I argue?

32. What is the specific gravity of the mixture of two fluids whose volumes are as 1 : 2, and specific gravity 3 : 4?

33. If weight = specific gravity × the volume, and the unit of length be 2 ft., what would be the unit of weight?

34. The specific gravity of salt water is 1·027; which is easier to swim in, fresh or salt water, and why?

35. The specific gravity of the air is about ·00125; what size balloon full of gas, whose specific gravity is ·0001, would lift 5 cwt.?

36. If very large jewels were weighed by spring balances, to whose advantage would it be if the air were very heavy?

37. Is the specific gravity of a man greater or less than that of water?

38. Has water always the same specific gravity?

39. A piece of copper weighs 1000 grains in air, 887 grains in water, and 910 grains in alcohol; find the specific gravity of alcohol. Give yourself 8·849 (S.G. of copper), the 887 grains, its weight in water, and find weight in air.

40. A uniform cylinder, when floating vertically, sinks to a depth of 4 in., to what depth will it sink in alcohol of specific gravity 0·79?

41. Give yourself the answer of 40, and find the specific gravity of alcohol.

42. A piece of cork (specific gravity ·24), containing 2 cub. ft., is just kept at the surface of the water by a weight placed on it; find the weight.

43. Two bodies, which weigh 5 lbs. and 4 lbs. in air, both weigh 3 lbs. in water; compare their specific gravities.

44. Specific gravity of sea water is 1·027; what proportion of fresh water must be added to a quantity of sea water that the specific gravity of the mixture may be 1·009?

45. A solid sphere floats in a fluid with three-fourths of its bulk above the surface; when another sphere half as large

again is attached to the first, the two spheres float at rest below the surface of the fluid. Show that the specific gravity of one sphere is 6 times that of the other.

46. A body whose weight is 6 lbs. weighs 3 lbs. and 4 lbs. respectively in two fluids; compare their specific gravities.

47. A body weighing 1 lb. weighs 14 oz. and 12 oz. in two fluids; what will it weigh in a mixture of two equal quantities of the fluids?

48. A symmetrical box, weighing 8 oz., with a weight on the top, floats just immersed in a fluid; how heavy must the weight be in order that when removed the box may float with only one-third immersed?

49. Find the weight of 54 cub. in. of copper, specific gravity 8·8.

50. If you forgot the weight of a cubic foot of water, but remembered that a cubic inch of gold (specific gravity 19·4) weighed $224\frac{1}{2}$ dwts., how would you find the weight of a cubic yard of cork, specific gravity ·24?

Miscellaneous Questions.

I.

1. Without multiplying them out, show that the cube of 4 = square of 8.

2. Find the cube root of 223648543, hence form the cube of 607.

3. Form the cube of 329, and prove your result by finding its cube root.

4. Divide 375 into four parts, so that the second is 2 greater than the first, and 2 less than the third, and $\frac{1}{2}$ the fourth.

5. Prove without dividing that 17325 is divisible by 7.

6. How many more times in a mile will the small wheel of a bicycle, diameter 16 in., revolve than the large, circumference 15 feet.

7. From what number shall I leave 1 more than its third, if I take away 8 more than its half.

8. What will it cost to cover a box on all six sides with tin

at 1s. a sq. ft., if it cost £1 to cover one double as long, double as broad, and of the same height, with tin 6d. a sq. ft. The length of the smaller box being treble the height and double the width.

II.

9. A man owns $\frac{3}{5}$ of a ship, and by selling $\frac{1}{2}$ his share for £100 he makes 2 per cent. profit; what did the whole ship cost?

10. The $\frac{1}{2}$ of a number is a 1000 greater than its $\frac{1}{5}$; what is the number?

11. Divide 623 into three parts, so that the 1st : 2nd as 4 : 7, and 2nd to the 3rd as 3 : 8.

12. Express in integers the ratios $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{6}$.

13. How many exact times is a line $\frac{1}{60}$ in. long contained in 1·03 yds.?

14. By what number must I divide a million, so that the quotient may equal the divisor, and leave a remainder 1999?

15. Express 3167 in the septenary scale, and square it.

16. Prove the working of 15.

III.

17. In three years, at a certain interest (compound), £625 produces £88·515625; what is the rate at which interest is calculated?

18. The difference between the discount and the interest on a certain sum of money for three months, interest calculated at 4 per cent., is 1s.; find the principal.

19. Give a common-sense interpretation of the formula

$$\text{rate} = \frac{100 \times \text{interest}}{\text{principal} \times \text{time}}.$$

20. A man purchases £10,000 3 per cent. stock at 92; if it rises to 102 and he cannot sell, what per cent. on his income does he lose?

21. It cost £1 more to paper a room with a paper 23 in. wide than with one 24 in. wide. If the length and breadth

are 16 and 14 feet, find the height, supposing that either paper cost 3s. a yard.

22. What is the solid content of a block of marble 4 ft. by 8 ft. by 6 ft., and what area of slabs 1 in. thick could be cut from it?

23. The discount on a sum due one year hence at £5 per cent. per annum interest is £15; find the sum.

24. What do you understand by the expression, 'the time varies inversely as the number of men employed'?

IV.

25. What fraction of £5 is the excess of $\frac{2}{3}$ of £2 over $(\frac{1}{2}-\frac{1}{3})$ of $\frac{3}{5}$ of £2, 10s.?

26. Why do you know, without trying, that £4, 9s. $1\frac{1}{2}$d. can be expressed by a terminating decimal of £5?

27. Find an expression in pounds alone equal to £4, 7s. $6\frac{1}{2}$d. which you could read at once as £ s. d.

28. Write down in £ s. d. the expression $\pounds 3\frac{4\frac{2\frac{1}{4}}{12}}{10}$.

29. Write down in £ s. d. the expression $\pounds 2\frac{1\frac{1}{6}}{5}$.

30. What pressure would a child have to exert on the long arm (1 mile) of a lever to raise 10 tons at the other end (arm 1 ft.)?

31. How many $\frac{1}{4}$ ft. per second does a railway train go which is travelling at 55 miles an hour?

32. If our units of measure be $7\frac{1}{2}$ ft. and 3 sec., what number will express 5 miles an hour?

V.

33. Explain carefully what you understand by losing 2 per cent. on a transaction in stocks.

34. If you purchase at $94\frac{1}{2}$, receive a half-yearly dividend

at 1s. a sq. ft., if it cost £1 to cover one double as long, double as broad, and of the same height, with tin 6d. a sq. ft. The length of the smaller box being treble the height and double the width.

II.

9. A man owns $\frac{3}{5}$ of a ship, and by selling $\frac{1}{2}$ his share for £100 he makes 2 per cent. profit; what did the whole ship cost?

10. The $\frac{1}{2}$ of a number is a 1000 greater than its $\frac{1}{5}$; what is the number?

11. Divide 623 into three parts, so that the 1st : 2nd as 4 : 7, and 2nd to the 3rd as 3 : 8.

12. Express in integers the ratios $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{6}$.

13. How many exact times is a line $\frac{1}{80}$ in. long contained in 1·03 yds.?

14. By what number must I divide a million, so that the quotient may equal the divisor, and leave a remainder 1999?

15. Express 3167 in the septenary scale, and square it.

16. Prove the working of 15.

III.

17. In three years, at a certain interest (compound), £625 produces £88·515625; what is the rate at which interest is calculated?

18. The difference between the discount and the interest on a certain sum of money for three months, interest calculated at 4 per cent., is 1s.; find the principal.

19. Give a common-sense interpretation of the formula

$$\text{rate} = \frac{100 \times \text{interest}}{\text{principal} \times \text{time}}.$$

20. A man purchases £10,000 3 per cent. stock at 92; if it rises to 102 and he cannot sell, what per cent. on his income does he lose?

21. It cost £1 more to paper a room with a paper 23 in. wide than with one 24 in. wide. If the length and breadth

and 10 respectively, which will fuse into a ball, containing 17 cub. ft., specific gravity 9·25.

48. What is the specific gravity of equal volumes of water and alcohol (specific gravity ·825) mixed together?

VII.

49. If after adding 14 galls. of water to 56 galls. of wine I sell at £1, 1s. a gallon and gain 20 per cent., what was the prime cost?

50. Upon what principal is discount calculated?

51. Show that the principal of a discount question is the amount of an interest one.

52. A tradesman buys books, 13 for 12, at a discount of 30 per cent. on the marked price, and sells them for cash at a discount of 3d. in the shilling, what is his gain per cent.?

53. If he gives 5 per cent. discount for 4 months' credit, 10 per cent. for 3 months, 15 per cent. for 2 months, and 20 per cent. for 1 month, and receives £100 in cash, and another £400 paid in four equal instalments at the end 1, 2, 3, and 4 months respectively, what did the books cost him?

54. A bankrupt owes A £100, B 200, C £300, and D £450; if his assets are only £42, what would B receive?

55. Find the square root of 14·4.

56. What do you understand by $\sqrt{3}$?

VIII.

57. Prove that $\sqrt{5} \times \sqrt{5} = 5$.

58. Reduce $\sqrt{\frac{3}{5}}$ to an expression with no root in the denominator.

59. Find a ratio equivalent to 3 : 5 whose terms differ by 70.

60. Find a ratio equivalent to 3 : 5 whose terms added together are 80.

61. Find a ratio equivalent to 3 : 5 whose terms multiplied together = 375.

62. Find the G. C. M. of 16851 and 39729.

63. How many lbs. of water whose temperature is 50° must be mixed with 100 lbs. whose temperature is 80° so as to reduce it to 60°?

64. It takes thirteen times as much heat to raise water 1° in temperature as it does to raise the same weight of mercury a degree, how many degrees would the heat in 10 lbs. of water, temperature 15°, raise 1 lb. of mercury, temperature 0°?

IX.

65. A circle 48 in. in diameter is cut out from one whose circumference is 30 ft.; find the area of the ring left.

66. If A and B each has £1000 in a business, but B does not come in with his at the beginning of the year; if out of £200 profits A has £130, when did B come into the business?

67. Find the equated time for discharging in one payment £202 due at end of 2 months, and £406 at the end of 3 months, and £412 at the end of 6 months, interest calculated at the rate of 6 per cent.

68. In 67 calculate present worth at any other rate than 6 per cent., and show that the result is different.

69. A owes £735, due 10 months hence; he pays £203 at the end of 3 months, and £309 at the end of 6 months, when was the remainder due? Interest calculated at the rate of 6 per cent.

70. Divide 312 into three parts, so that five times the second is less than the third by the first, the first being 6.

71. Could you divide 312, as in 70, with another number than 6 for first?

72. Convert 357 florins 40 kreuzers (1 florin = 60 kreuzers) into Russian roubles and copecks (1 rouble = 100 copecks), 1 florin = 2·16 francs, and 4 francs = 1 rouble.

X.

73. Four circles, each 1 in. in diameter, are so placed that two of them touch two of the others, and the remaining two

both touch the three others; find the area of the rhombus whose angles are at the four centres.

74. Prove from first principles of value by position, that 3542 is divisible by 7 in the octonary scale.

75. Find the fifth power of 37.

76. Find the fifth root of answer of 75.

77. In how many years at 5 per cent. simple interest will £100 double itself?

78. Why can't you find by pure arithmetic the number of years money doubles itself in by Compound Interest?

79. Place a weight of 1 lb. so as to overcome the upward pressure of a force of 10 lbs. which is exerted 3 ft. from the fulcrum.

80. Transform 625 into the quinary scale, and extract its square root.

XI.

81. How much water per gallon must I mix with whisky of 50 per cent. above proof to reduce it to 12 per cent. above proof, water being 100 below proof, and proof spirits being half water and half pure alcohol?

82. Multiply £3, 7s. 8½d. by $\frac{\text{1 lb. 10 dwts.}}{\text{2 oz. 10 dwts.}}$.

83. Divide the result of 82 by $\frac{\text{1s.}}{\text{£1}}$.

84. If A can do in 5 days what B can do in 4 days and C in 3 days, how long would they all three, working together, take to do what A alone could do in 30 days?

85. Define proportion. What number must I take away from each of the four numbers 3, 4, 5, 7, so that they may be in proportion?

86. A grocer buys tea wholesale at the rate of 2s. a pound, and in weighing it out to his customers uses a pair of scales the arms of which are 16 in. and 15 in.; at what price must he propose to sell it per pound so as to make a real profit of 20 per cent.?

87. Find the area of the base of a cylinder whose height is 10 ft. and which weighs 1 ton (specific gravity 4).

88. What is the moment of a force of 10 lbs. acting 4 ft. from a fulcrum?

XII.

89. Why does a balloon ascend in the air as cork does in water?

90. A lifeboat weighing 1 ton displaces 1 cubic fathom of water; what is its sustaining power? (Specific gravity of sea. water = 1·027.)

91. Two cylindrical rods, one of platina (specific gravity 21·53), the other of silver (specific gravity 10·5), having the same diameter, are joined with their axes in the same straight line. If they balance at their point of junction, and the length of the silver rod is 10 in., what is the length of the other rod?

92. Give yourself the answer of 91, and find the specific gravity of platinum.

93. Equal vôlumes of two bodies weigh 4 lbs. and 9 lbs.; compare their specific gravities.

94. How many bricks 8 in. by $2\frac{1}{2}$ by $3\frac{1}{2}$ will be required to face the four sides of a cubical building, each side 20 ft., the facing being 7 in. thick (don't forget the corners)?

95. If the surface of the exposed side of a glazed brick be 9×4 in., how much will it cost to face with them a kitchen 30 ft. × 20 ft. and 12 ft. high at £2 a hundred, ignoring their thickness?

96. Divide 770 into four parts that the same digits will express them in the senary, septenary, octonary, and nonary respectively. What is the next number that could be so divided integrally?

HINTS AND ANSWERS.

HINTS AND ANSWERS.

PART I.

CHAPTER I.

1. Value by position.

2. Since the value of a figure in a whole number depends on its position from the right, any ciphers—in themselves valueless—cannot affect the position of the other figures, nor therefore alter their value. Similarly, in a decimal fraction the figures gain their value by their position with respect to the decimal or unit point which is to their left; any cipher—in itself valueless—added to their right will not affect their position, nor therefore alter their value.

3. If we add on a 7 to 453, we multiply the 453 by 10 and add 7 to the product.

4. If we insert a 3 between the 4 and the 5 in 4572, we multiply the 4000 by 10 and add on 3000.

5. 360360 will divide by 2, 5, 10, because the last figure will.

,,	,,	4, because the last two figures will.
,,	,,	8, ,, last three ,,
,,	,,	3, 9, because $3 + 6 + 3 + 6 = 18$.
,,	,,	7, 11, and 13, because number $= 1001 \times 360$, and $1001 = 7 \times 11 \times 13$.
,,	,,	6 and 12, because it is divisible by 4 and 3.
,,	,,	14, ,, ,, 7 and 2.
,,	,,	15, ,, ,, 5 and 3.

6. Since $3+4+2+2+1+3=15$,

$$342213 = \text{some nines} + 6.$$

Since $(3+2+4)-(1+2+3)=3$,

$$342213 = \text{some elevens} + 3.$$

Since $213 = 26 \times 8 + 5$;

$$\therefore\ 342213 = \text{some eights} + 5.$$

7. Since $2+5+3+4+1+7=22=3\times 7+1$,

$$253417 = \text{some threes} + 1,$$

and since $17 = 4 \times 4 + 1$,

$$253417 = \text{some fours} + 1;$$

$\therefore$ if we divide by 3 first, the remainder must either be 1 or 4 or 7 or 10; but if we divide by 4 first, the remainder must be 1 or 5 or 9;

$$\therefore\ 253417 = \text{some twelves} + 1.$$

If we divide by 2 first, the remainder must be 1, 3, 5;

$$\therefore\ 253417 = \text{some sixes} + 1.$$

8.
$$\begin{array}{r} 7\underline{)100000} \\ 0142857 \\ \underline{1326451} \end{array}$$

$$\begin{array}{lll} 2 \times 1 = & & 2, \\ 7 \times 3 = \text{some} & \text{sevens} & +0, \\ 3 \times 2 = & & +6, \\ 9 \times 6 = & \quad ,, & +5, \\ 4 \times 4 = & \quad ,, & +2; \\ \therefore\ 49372 = \text{some} & \text{sevens} & +1. \end{array}$$

9. $11,\ 10+1,\ 12-1,\ 11\times 1,\ 22\div 2,\ \frac{33}{3},\ \sqrt{121}$.

10. 347 means 7 ones, 4 tens, and 3 ten tens or 3 hundreds.
11. 347 (octonary) means 7 ones, 4 eights, and 3 sixty-fours.
12. 347 (octonary) = 231 (denary).
13. 1411.
14. 2707.
15. 3637.
16. 9335.
17. 2101313.
18. 9668.
19. 111.

20. 1101111.

21. 45.

22. 39.

23. ·475 means 4 tenths, 7 hundredths, and 5 thousandths.

24. By calculating the value of figures from the left instead of from the right.

25. Three hundred and twelve hundredths.

26. 47 thousand 6 hundred and 81 thousandths.

27. 3 thousand 7 hundred and 54 millionths.

28. One thousand and one tenths.

29. ·001020.

30. 14·037.

31. ·029 + ·0219 or ·0509.

32. 10·04027.

33. ·346 in the septenary scale means 3 sevenths, 4 forty-ninths, and 6 three hundred and forty-thirds $\left(\frac{6}{7 \times 7 \times 7}\right)$.

34. ·14.

35. ·592.

36. ·3.

37. ·921875.

38. 423·5.

39. 19·344.

40. 1433·142.

41.
$$\begin{array}{r|rrrrrrrr} 17) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & 0 & 0 & 5 & 8 & 8 & 2 & 3 & 5 \\ & & 1 & 10 & 15 & 14 & 4 & 6 & 9 & 5; \end{array}$$

$\therefore$ remainder $= 1 + 3 + 15 + 16 + 8 + 1 + 12 + 3 = 59$, or $3 \times 17 + 8$, or remainder $= 8$.

42. 4, 1, 6.

43. 5, 8, 4.

44. 57, 98, 23.

45. It is neither too large nor too small, corresponds to the fingers, etc., on the two hands, and is now the basis of the language of numbers. Its disadvantage is, that it has so few integral aliquot parts, and especially no integral quarter.

46. Let 47 be any number expressed with two figures.

Thus, $47 = 4 \times 9 + 4 + 7$,
and $74 = 7 \times 9 + 7 + 4$;
$\therefore$ the difference must be 3×9.

47. The proof of 46 will suggest this answer.

48. A digit means finger, and ten was doubtless taken as the ordinary radix because of the ten fingers with which men ordinarily counted and probably expressed numbers.

49. Between the 0 and the 1.

50.

327131	= some	twos and fives and tens	+ 1.
	= ,,	fours	+ 3.
	= ,,	eights	+ 3.
	= ,,	threes	+ 2.
	= ,,	nines	+ 8.
	= ,,	elevens	+ 2.
	= ,,	sixes	+ 5.
	= ,,	twelves	+ 11.
	= ,,	sevens	+ 0.
	= ,,	fourteens	+ 7.
	= ,,	fifteens	+ 11.
	= ,,	thirteens	+ 12.

CHAPTER II.

1. 307737.

2. I add to the minuend one of the next denomination in the form of the denomination in which I am subtracting, and in the form of one of the higher denomination to the subtrahend. Thus in $347 - 164$ I add to the 347 one hundred in the form of ten tens to the minuend, and in the form of one hundred to the subtrahend. Separating the figures by commas

the minuend is	3, 14, 7,
and the subtrahend	2, 6, 4;
hence the difference	1 8 3.

3. 94841.

4. 106775.

5. 3689243.

6. 136.

7. 57608.

8. 9964.

9. 1101 with remainder $8 \times 35 + 5 \times 5 + 1 = 306$.

10. $347121 =$ some number of fives $+ 1$;

$\therefore$ the remainder is 1 or 6 or 11 or 16, etc. (i.)

Again, $347121 =$ some number of sevens $+ 5$;

$\therefore$ the remainder is 5 or 12 or 19 or 26, etc. (ii.)

Again, 347121 is some number of nines $+ 0$;

$\therefore$ the remainder is 0 or 9 or 18 or 27, etc. (iii.)

From ii. and iii. 347121 is some number of sixty-threes $+ 54$;

$\therefore$ the remainder is 117 or 180 or 243 or 306;

and from i. and iii. 347121 is some number of forty-fives $+ 36$;

$\therefore$ the remainder is 81 or 126 or 171 or 216, or 261 or 306;

$\therefore$ the remainder must be 306.

11. See par. 9.

12. In finding out the remainder after dividing by 8 without division, you do not take into account any of the figures beyond the last three.

You therefore only test the accuracy of the last three figures.

13.

1	2	3	4	5	6	7	10
2	4	6	10	12	14	16	20
3	6	11	14	17	22	25	30
4	10	14	20	24	30	34	40
5	12	17	24	31	36	43	50
6	14	22	30	36	44	52	60
7	16	25	34	43	52	61	70
8 or 10	20	30	40	50	60	70	100
9 ,, 11	22	33	44	55	66	77	110
10 ,, 12	24	36	50	62	74	106	120
11 ,, 13	26	41	54	67	102	115	130
12 ,, 14	30	44	60	74	110	124	140

(i.) 171633167. (ii.) 5241234.

14. Since $4 + 3 + 3 + 6 + 2 = 18 = 6 \times 3$, therefore 43362 (septenary) is divisible by 6.

15. Since $(2 + 3 + 4) - (4 + 5) = 0$, 45342 is divisible by 10 (denary), *i.e.* 11 (nonary).

16. $357357 = 357 \times 1001$ and 1001 nonary, since $(1+0)-(0+1)=0$ is divisible by 10 (denary) *i.e.* 11 nonary, and therefore by 5.

17. 8151482999628. $14400 = 12 \times 12 \times 100$,
and $172800000 = 12 \times 144 \times 100000$.

18. 7189121628. $152412 = 144000 + 8400 + 12$. Hence to get first line multiply multiplicand by 12; for second, multiply first line by 7; and for third, multiply first line by 12.

19. 62839532349. $1332221 = 1331000 + 1210 + 11$, and $1331 = 11 \times 11 \times 11$.

20. 26.

21. 9, $10-1$, 3×3, $18 \div 2$, etc.

22. £4, 13s. $4\frac{1}{2}$d.

23. £1, 13s. $4\frac{1}{2}$d.

24. £10, 15s. 2d., or double £5, 7s. 7d.

25. 15 minutes.

26. 60 miles. I have to divide 2 hours by 2 minutes.

27. A has £137, B £120, C £110. Subtract what A and B have more than C, and divide remainder by 3.

28. They meet $25\frac{1}{7}$ miles from London, and B is $6\frac{1}{2}$ miles from London when A arrives at Brighton.

29. $533\frac{53}{453}$

453 times
1 = 453.
2 = 1350.
3 = 2243.
4 = 3140.
5 = 4033.

30. In the binary though we have only two symbols, viz. 0 and 1, it requires a great many figures to express even a small number.

The duodenary has the advantage of having two more integral aliquot parts than the denary; but it is generally considered too large a root, and, were it adopted, we should have to frame a language to suit it.

31. 63 (denary) = 111111 (binary), 107 (denary) = 8e (duodenary), for which we have no appropriate expression, nor even a single numerical symbol for eleven.

32. Each child had 10s., woman 15s., and man £2, 3s. 9d.

33. Each child had £1, 7s., each woman £1, 12s., and each man £1, 19s.

From £99, 9s. subtract the men's excess, viz. 20×12s., and the women's excess, viz. 15×5s., and treat them all as children.

34. £21, 19s. This is very easy.

35. You can easily find A's rate and thence B's. A's $= 10$; B's 15 till 12·20.

36. A has to stop 1 hr. and 10 min.

37. B's portion is £24.

Since A has £10 more, and C £5 more, and D £3 less than B, subtract from £108, £$(10 + 5 - 3)$, and divide remainder by 4.

38. £108. This is only asking for one of the elements of 37, the £24 (B's portion) being known.

39. B has £5 less than C.

40. C has £20 and D £25.

C's share is evidently one-fifth of 100, and B's and D's together $=$ £40.

41. If E's $= 1\frac{1}{5}$, or $\frac{6}{5}$ of D, E has £30.

42. See answer to 40.

43. $(7+4)+(7-4) = 2 \times 7$,

$(5+3)+(5-3) = 2 \times 5$.

44. $(7+4)-(7-4) = 4 \times 2$,

$(1\text{s.} + 2\text{d.}) - (1\text{s.} - 2\text{d.}) = (14 - 10)\text{d.} = 2 \times 2\text{d.}$

45. The middle part is evidently one-third of 99, or 33. To find the others $(99 - 33 - 10) \times \frac{1}{2} = 28$, and 38.

46. 83t18 (undenary), or 122350 (denary).

47. 1327 and remainder 16 (both expressed in octonary), or 727 and remainder 14 (denary).

48. 1202 and 7 remainder, or 893 and 7 remainder (denary).

After dividing by 4 we get three units, and after dividing by 7 we get 1 four; $\therefore$ remainder $= 7$.

49. $3751 = 11 \times 11 \times 31$ denary $= 14 \times 14 \times 43$ (septenary) $=$ 13636 (septenary).

50. $1034\frac{153}{343}$.

CHAPTER III.

1. 122 lbs. 8 oz. 18 dwts. 8 grs.

2. 191 lbs. 4 oz. 17 dwts. 12 grs.

3. 3 lbs.

4. 1 sc. = 20 grs., and 1 dwt. = 24 grs.; ∴ 6 sc. = 5 dwts. Again, 1 dr. = 60 grs., and 1 dwt. = 24 grs.; ∴ 2 drs. = 5 dwts.

5. 17 dwts.

6. 144.

7. 3640 lbs. tea, 7560 lbs. coffee, or 2457 lbs. of both tea and coffee.

8. 863 plots.

9. (9×1461) miles, $1461 = 4 \times 365 + 1$ (leap year).

10. £41008, 10s. $3\frac{3}{4}$d.

11. 182 ac. 0 rd. 29 po. 25 yds. 6 ft.

12. This is asking for an element in 9. Ans. $3\frac{1}{2}$.

13. Wednesday, July 4, 1855. The remainder, after dividing by six, will give the day of the week and number of weeks, and this divided by 52 will give the year. $29 \times 365 + 8$ would tell how many copies have been published since October 20, 1855, if it had been published on Sundays; hence, one-seventh of the number must be subtracted from it, and the remainder, after subtracting this difference from 9173, will give the number published on October 20, 1855; hence the few remaining days can be easily calculated.

14. Since $504 = 4 \times 7 \times 18$, there are both an exact number of periods with the extra day for leap year, and an exact number of sevens; so it matters not on what day of the week the first day of the first year fell; hence answer 26298, see 16.

15. From October 19, 1884, to February 29, 1880, was 7×242 days; hence February 29, 1880, was a Sunday. From October 19, 1884, to September 17, $1886 = 365 \times 2 - 32$; hence there were altogether 341 weeks and 5 days, or 342 Sundays. March 1, 8, 15, 22, 29.

16. 26351 or 26350, according as the first Sunday in the first year fell in or out of the first two days of the years. It also might depend on which of the first three years was leap year, and

whether the 505 years contained a multiple of 4, which was not a leap year.

17. 11 o'clock a.m.

18. 12 hrs. 17′ p.m.

19. (1) 44 minutes back; (2) 44 minutes forward.

20. $8° 52' 30''$. 60 seconds = 1 minute, 60 minutes = 1 degree.

21. 99800.

22. $\dot{2}35$ and 221 tenths of a thousandth of an inch over.

23. 7 lbs. 7 oz. 14 dwts. 14 grs.

24. £2415, 2s. 3d.

25. $(70 \times 2\frac{1}{2} - \frac{1}{12}$ of $2\frac{1}{2})$ minutes forward. 2 hrs. $54' \, 47\frac{1}{2}''$.

26. $54'' \frac{412}{959}$ seconds a day; there are 160 days less 4 hours.

27. 11 hrs. $39' \, 30''$ a.m.

28. See 27.

29. A clock was losing regularly $1\frac{1}{2}$ minutes each day, but on April 3 its regulator was touched, which made it gain $\frac{1}{2}$ a minute a day. On June 3 it was found to indicate 11 hrs. $39' \, 30''$ at noon; when did it begin to lose?

30. A clock which was correct on February 29 began to lose regularly. On April 3 its regulator was touched, so that it gained $\frac{1}{2}$ a minute a day. On June 3 it pointed to 11 hrs. $39' \, 30''$ at noon; how much a day was it losing between February 29 and April 3?

31. A clock which was correct on February 29 began to lose $1\frac{1}{2}$ minutes a day. On April 3 its regulator was touched, so that it gained $\frac{1}{2}$ a minute a day. On what day at noon did it point to 11 hrs. $39' \, 30''$?

32. A clock which was correct on February 29 began to lose $1\frac{1}{2}$ minutes a day. On April 3 its regulator was touched, so that on June 3 at noon it pointed to 11 hrs. $39' \, 30''$. What effect had this touch of the regulator upon it?

33. $$£\frac{4257 \times 52 \times 4 \times 3}{2 \times 16 \times 12 \times 20} \text{ or } £115,\ 5s.\ 10\tfrac{1}{2}d.$$

34. He receives 17s. $0\frac{1}{8}$d. and pays 1s. 2d.; hence, profits 15s. $10\frac{1}{8}$d. Apoth. oz. = 480 grs.

35. Tea cost him 920s. 10d., he receives 975s.; ∴ his profit is 54s. 2d.

36. It is a number of weeks + 4 days; ∴ February 29, 1836, fell on a Monday.

37. Since they approach one another at double the rate at which they are going, they must meet at intervals of half of what they start at.

38. 12.

39. He purchases £1002 worth, which he marks at £1336. The £300 he receives ready money represents as many shillings as there are 10d. in £300, which = £360. He therefore books £976; of this he loses 976s., or £48, 16s.; ∴ he receives £300 + 927, 4s., or £1227, 4s.; that is, his profits are £225, 4s.

40. See 39.

41. A tradesman purchases £1002 worth of goods, which he marks at a uniform profit. He receives £300 in ready money, when he gives a discount 2d. in the shilling. On his book transactions he loses 1s. in the £, and altogether makes £225, 4s. profit. At what profit did he mark his goods?

42. A tradesman buys £1002 worth of goods, and marks them at the uniform profit of 4d. in the shilling. He receives £300 for ready money transactions, in which he gives a certain reduction. On his book transactions he loses 1s. in the pound, and altogether he makes £225, 4s. What reduction does he make for ready money?

43. A tradesman buys £1002 worth of goods, and marks them at the uniform rate of 4d. in the shilling. He allows 2d. in the shilling for ready money. He loses 1s. in bad debts, but makes on the whole £225, 4s. How much did he receive as ready money?

44. A tradesman buys £1002 worth of goods. He marks them at a uniform profit of 4d. in the shilling, allowing 2d. in the shilling on the marked price for ready money. He receives £300 in ready money payments; but on the whole transaction he only makes £225, 4s. What were his bad debts?

45. A makes £409. B, £312, 16s. A's gain £96, 4s.

46. You could not find the original price; as being the same, it cannot affect the difference.

47–50. See 45.

CHAPTER IV.

1. Of a £, $\frac{1}{2}$ = 10s.; $\frac{1}{3}$ = 6s. 8d.; $\frac{1}{4}$ = 5s.; $\frac{1}{5}$ = 4s.; $\frac{1}{6}$ = 3s. 4d.; $\frac{1}{8}$ = 2s. 6d.; $\frac{1}{10}$ = 2s.; $\frac{1}{12}$ = 1s. 8d.; $\frac{1}{15}$ = 1s. 4d.; $\frac{1}{16}$ = 1s. 3d.; $\frac{1}{20}$ = 1s.; $\frac{1}{24}$ = 10d.; $\frac{1}{30}$ = 8d.; $\frac{1}{32}$ = 7$\frac{1}{2}$d.; $\frac{1}{40}$ = 6d.; $\frac{1}{48}$ = 5d.; $\frac{1}{60}$ = 4d.; $\frac{1}{64}$ = 3$\frac{3}{4}$d.; $\frac{1}{80}$ = 3d.; $\frac{1}{96}$ = 2$\frac{1}{2}$d.; $\frac{1}{120}$ = 2d.; $\frac{1}{160}$ = 1$\frac{1}{2}$d.; $\frac{1}{240}$ = 1d.; $\frac{1}{480}$ = $\frac{1}{2}$d.; $\frac{1}{960}$ = $\frac{1}{4}$d.

2. Of 6s. 8d., $\frac{1}{2}$ = 3s. 4d.; $\frac{1}{4}$ = 1s. 8d.; $\frac{1}{5}$ = 1s. 4d.; $\frac{1}{8}$ = 10d.; $\frac{1}{10}$ = 8d.; $\frac{1}{16}$ = 5d.; $\frac{1}{20}$ = 4d.; $\frac{1}{40}$ = 2d.; $\frac{1}{64}$ = 1$\frac{1}{4}$d.; $\frac{1}{80}$ = 1d.; $\frac{1}{160}$ = $\frac{1}{2}$d.; $\frac{1}{240}$ = $\frac{1}{3}$d.

3. Of 1 ton, $\frac{1}{2}$ = 10 cwt.; $\frac{1}{4}$ = 5 cwt.; $\frac{1}{5}$ = 4 cwt.; $\frac{1}{8}$ = 2 cwt. 2 qrs.; $\frac{1}{10}$ = 2 cwt.; $\frac{1}{16}$ = 1 cwt. 1 qr.; $\frac{1}{20}$ = 1 cwt.; $\frac{1}{40}$ = 2 qrs.

4. Of 1 acre, $\frac{1}{2}$ = 2 ro.; $\frac{1}{4}$ = 1 ro.; $\frac{1}{8}$ = 20 po.; $\frac{1}{10}$ = 16 po.; $\frac{1}{20}$ = 8 po.; $\frac{1}{22}$ = 220 yds.; $\frac{1}{40}$ = 121 yds.; $\frac{1}{44}$ = 110 yds.; $\frac{1}{55}$ = 88 yds.; $\frac{1}{88}$ = 55 yds.; $\frac{1}{110}$ = 44 yds.; $\frac{1}{121}$ = 40 yds.; $\frac{1}{242}$ = 20 yds.; $\frac{1}{440}$ = 11 yds.

5. £6342, 14s. 8$\frac{3}{4}$d.

6. In a half there are 12 twenty-fourths; in a third 8; in a quarter 6, in a sixth 4, in an eighth 3, in a twelfth 2. Ans. 65 twenty-fourths, or 2 and 17 twenty-fourths.

7. £2203, 2s. 6d.

8. £2863, 6s. 0d.

9. £26, 13s. 0d. The second aliquot part is used to save Long Division.

10. 2s. is $\frac{1}{10}$ of £1, and 2$\frac{2}{5}$d. is $\frac{1}{10}$ of 2s.

11. £22, 12s. 3$\frac{1}{3}$d.

12. 7$\frac{121}{315}$d.

13. 3 fur. 32 po. 2 yds. 1$\frac{1}{3}$ ft.

14. 4 sq. yds. 7$\frac{9}{28}$ ft.

15. 2404$\frac{2}{3}$.

16. 343$\frac{2}{33}$.

17. 1130$\frac{27}{70}$.

18. 341$\frac{29}{84}$.

19. 3 ro. 8 po. 12 yds. 0 ft. 130 in.

20. 16s. 0$\frac{78409}{130680}$d.

21. 2 × 3 × 14 × 2 × 3 × 3.

22. Since there are an even number of odd digits, all to multiply

by odd numbers, 7, 49, etc., and the sum of two odd numbers is even, hence the sum of four *odd* numbers must be. We could also have seen it thus. Since $3+3+1+5=12$ (denary) $=6\times 2$; $\therefore$ 3315 (septenary) will divide by 6 (radix $7-1$), and therefore by 2.

23. $12\times 2\times 2\times 2\times 2\times 2\times 2\times 5$.

24. $(4+4)-(7+1)=0$; $\therefore$ the number will divide by 10 (denary) or radix $+1$. Again, since $4+4+7+1=8\times 2$; $\therefore$ the number will divide by 8 (radix -1).

25. 16904 twelfths, or 1690 units and 4 twelfths written 1690·4.

26. $143\frac{164}{528}$.

27. 26 lbs. 11 oz. 3 dwt. 4 grs.

28. 10500.

29. $\frac{1}{30}$.

30. $\frac{1}{99}$ in., and $\frac{2}{99}$ in. This instrument is called a Vernier from the name of the Italian who invented it.

31. 12.

32. 868645.

33. $471321=$ some number of twos, fives, or tens, and 1; some fours and 1; some eights and 1.

Some threes and some nines and 0. Some sixes and 3 (since if we divide by 3 first, the remainder must be 0 or 3; and if we divide by 2 first, the remainder must be, 1, 3, or 5).

Some twelves and 9, some elevens and 4, some sevens and 4.

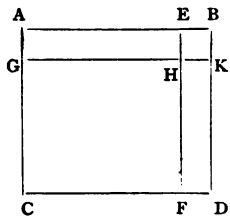
34. Since $792=11\times 8\times 9$, and 133056 is exactly divisible by all these three; $\therefore$ the number is divisible by their product.

35. Let 36, 37, 38, 39 be any four consecutive numbers of which 37 is one;

$$\begin{aligned}\therefore 36\times 37\times 38\times 39 &= 4\times 9\times 37\times 2\times 19\times 3\times 13\\ &= (1\times 2\times 3\times 4)\times 9\times 37\times 19\times 13.\end{aligned}$$

In four consecutive numbers one must divide by 4, and 2 by 2, and 1 by 3. Hence in any four there must appear as factors 4, 3, 2, 1.

36. Any three consecutive numbers, of which the number is one, must contain a multiple of 3, and therefore the number must be either this number or just before it or just after it,—*that is*, a multiple of 3 or one more or one less.



37. Every number is either even or odd. The square of any even number (as 38) $= 2 \times 19 \times 2 \times 19$, which is a multiple of 4. Any odd number may be expressed as one more than an even number. Let $11 =$ any odd number, and let us write this $2 \times 5 + 1$. In the figures above let AE represent 2×5, and EB 1. Then AD, the square on AB, is made up of the four figures, GF, AH, HD, and EK, of which AH and HD are equal, or 2AH.

Now $GF = 2 \times 5 \times 2 \times 5$, or some multiple of 4, and $2AH = 2 \cdot 2 \times 5 \times 1$, or some multiple of 4, and $EK = 1$;

$\therefore$ the four figures or square of $AB =$ some multiple of 4 square feet $+ 1$.

38. 32, 37, 42, 47, 52, 57, 62, 67.

39. 107, 118, 129, 140, 151.

40. Had he performed the whole journey, he would have received £35, but he only receives £2; therefore of the remainder, £33, he first earns £22 and loses it again in failing over that which would have given him the remainder, £11. Therefore, to find when he broke down, calculate the £11 from the end, viz. £7 for the last 24 miles (series), £1 + £2 for the riding and driving of the 4th series, and £1 more or 2 miles of the walking in which he failed. Therefore altogether he performed 3 series and 6 miles of the fourth walk, failing in 2 miles of the fourth walk, all the fourth riding and driving, and all the fifth series.

41. After we have subtracted the amount of money he received from £35, divide the remainder by 3, and he is as many miles, walking or riding or driving, from home as this money would pay him.

42. £35 − 14 = £21, and £21 ÷ 3 = £7; $\therefore$ he is £7 from the end—that is, all the driving or 40 miles, for which he would receive £5, and 8 miles (£2) of the riding—that is, 48 miles from the end.

43. For riding he received £10 and for walking £20, but as he could not drive, he had to pay £10. $\therefore$ Ans. £20.

44. He walked, rode, and drove $\frac{2}{3}$ of 40 miles, or $26\frac{2}{3}$ miles.

45. The one clock has gained $52'$ exactly, therefore the right time at 9 hrs. $52'$ Christmas Eve is 9 a.m.

The other clock has lost $42''$ a week, or $42'' \times 52$ in the year; $\therefore$ *i.e.* $36'\ 24''$, or the clock makes 8 hrs. $23'\ 36''$.

46. By giving 4d. and receiving 3d. back, I can pay 1d. Hence if I can pay 1d., I can pay any number of pence.

47. Reduce the number to the ternary (3) scale, thus—

$$\begin{array}{r l}
3\underline{)38} & \\
3\underline{)12} & \ldots\ 2 \\
3\underline{)4} & \ldots\ 0 \\
1 & \ldots\ 1
\end{array}$$

$\therefore$ $38 = 1102$, but to weigh the 2 pounds the 3 lbs. must be put into the one scale and the 1 in the other. This subject will be discussed farther on. For 25, which is $= 221$. If we add 1 three pound on to the 2 three pounds, we make them 3 three pounds or 1 nine pound, and this nine pound added on to the 2 nines, make 1 twenty-seven. Hence to weigh 25 lbs., take the twenty-seven and the one and put the three into the other scale.

48. $4\frac{3}{4}$ quarters.

49. Better by 2s. 11d.

CHAPTER V.

1. Since decimals can be added and subtracted without Reduction, it is better to add and subtract fractions in a decimal form; but since in Multiplication and Division cancelling cannot be resorted to in decimals, it is better to multiply and divide fractions in the form of Vulgar Fractions.

2. If you multiply the numerator you take the fraction so many times, or in other words multiply it; but if you multiply the denominator of a fraction by any number you divide the fraction by that number; hence, if you multiply both numerator *and* denominator by the same number you both multiply and

divide the fraction by the same number, which can have no effect upon the fraction.

Again, to multiply numerator and denominator by same number, as 7, is to take the fraction $\frac{7}{7}$ or once—which can only give us the original value of the fraction.

Again, if I multiply the denominator of a fraction by any number, as 4, I change it into a denomination of which there are four of the former; hence there must be four times as many, which is found by multiplying the numerator by 4.

3. An even number is one that will divide by 2 without remainder. An odd number is one that leaves a remainder 1 after being divided by 2. Hence, in two odd numbers, if added or subtracted, there will be 2 ones either to add on or to disappear from the other twos; hence the results must be even.

4. If a number will divide by 3 it is a multiple of 3; if it will not divide by 3 it must leave a remainder of either 1 or 2. Let any number, after being divided by 3, leave a remainder 1, the next would leave a remainder 2, and the next would divide by 3.

Since in any three consecutive numbers there is a factor 3 and a factor 2, hence the product of any three consecutive numbers is divisible by 1, 2, 3, or 6.

5. A prime number is one that has no factors but itself and unity—

13, 17, 19, 23, 29, 31, 37, 41, 43, 47,
53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

6. If a number is a common measure of two numbers it is also a measure of the sum or difference of any multiples of those numbers.

7. This is proved in the proof of G. C. M.

8. Let 7×31 and 9×31 be any two numbers; 31 is their G. C. M., then their L. C. M. $= 31 \times 7 \times 9$;

but $31 \times 7 \times 9 = \dfrac{131 \times 71 \times 31 \times 9}{31}$, or the two numbers multiplied together and divided by their G. C. M.

9. G. C. M. of 1703 and $2227 = 131$;

$\therefore$ L. C. M. 1703×17, or 28951.

10. G. C. M. of 1963, 2567, and 3171 is 151;

$\therefore$ L. C. M. $= 151 \times 13 \times 17 \times 21$.

11. 47.

12. 391.

13. $3 \times 3 \times 5 \times 5 \times 11$; $2 \times 2 \times 2 \times 3 \times 3 \times 11 \times 47$; $2 \times 13 \times 37$.

14. $\dfrac{2408}{17 \times 151 \times 159}$.

15. $\dfrac{6086}{3 \times 5 \times 7 \times 3 \times 11}$.

16. $5\frac{13}{30}$.

17. $\dfrac{775 \times 2}{2625}$ or $\dfrac{62}{105}$; $\dfrac{76\frac{1}{2}}{160\frac{2}{3}}$ or $\dfrac{459}{964}$; $\dfrac{10327}{13803}$.

These results are abstract numbers.

18. This really is finding the L. C. M. of $2\frac{1}{4}$, $3\frac{1}{3}$, $4\frac{1}{6}$, and $6\frac{3}{4}$ shillings, which = £67, 10s.

19. $3\dfrac{4}{5\frac{1}{6}} \div 2\dfrac{3}{4\frac{1}{5}}$ or $1\dfrac{230}{589}$.

20. G. C. M. $\frac{5}{9}$. L. C. M. $= 5 \times 79 \times 751$.

21. Because, if a number be contained in another it is manifest any measure of that number must be; *e.g.* since 3 is contained in 12 four times, 3 eighths must be contained four times, eight times, or thirty-two times. The denominator of the G. C. M. of fractions must be the L. C. M. of the denominators of the fractions.

22. The brandy left is $\dfrac{3 \times 3 \times 3 \times 3}{4 \times 4 \times 4 \times 4}$, or $\dfrac{81}{256}$ of cask.

23. 680.

24. 15153.

25. $29\frac{7}{16}$.

26. $\dfrac{3}{5} = \dfrac{108}{5 \times 36}$.

27. $4 = \dfrac{4}{1} = \dfrac{1}{\frac{1}{4}} = \dfrac{7}{\frac{7}{4}}$.

28. $4\dfrac{1}{7\frac{1}{2}} = 4\dfrac{2}{15} = \dfrac{62}{15} = \dfrac{\frac{62}{3}}{5} = \dfrac{\frac{62 \times 20}{3}}{100} = \dfrac{413\frac{1}{3}}{100}$.

29. $\dfrac{\dfrac{38 \times 100}{83}}{100} = \dfrac{45\dfrac{65}{83}}{100}.$

30. $\dfrac{4s.\ 6d.}{5s.\ 8d.} = \dfrac{4\frac{1}{2}}{5\frac{2}{3}} = \dfrac{27}{34} = \dfrac{1}{\frac{34}{27}} = \dfrac{4 \text{ lbs.}}{\frac{34 \times 4}{27} \text{ lbs.}} = \dfrac{4 \text{ lbs.}}{5\frac{1}{27} \text{ lbs.}}$

31. $\frac{1}{4}$.

32. 3904.

33. $\dfrac{4815}{379}$ seconds $= \dfrac{4815}{379 \times 60 \times 60}$ hours $= \dfrac{107}{379 \times 80}$ hours.

34. $\dfrac{3641}{411}$ sq. ft. $= \dfrac{3641 \times 4}{411 \times 9 \times 121}$ sq. po. $= \dfrac{1324}{411 \times 99}$ sq. po.

35. $10\frac{5}{12}$ months.

36. See 35.

37. See 35.

38. Not knowing the increase, in expressing his expenditure the second and following years you would have to perform operations on an unknown quantity—which is not the province of arithmetic.

39. $\frac{94}{215}$.

40. $38\frac{1}{4}$.

41. See 40.

42. See 40.

43. $\frac{200}{273}$.

44. G. C. M. 27. Fr. $= \frac{16}{18}$.

45. G. C. M. = 2143. Fr. $= \frac{21}{23}$.

46. G. C. M. = 14. L. C. M. = 32104.

47. 5013.

48. $5\frac{43268}{61759}$.

49. 16.

50. $2\frac{23}{210}$ (denary) = 2·1393 etc. (duodenary).

CHAPTER VI.

1. Miles and hours. 50 miles an hour $= 73\frac{1}{3}$ ft. a second.

2. 15 horses for two days costing £2, is the same as 30 donkeys for 24 hours costing 40s.

3. 3 men earning £10 in 8 weeks, is the same as 12 boys earning 2400 pence in 576 hours.

4. Take as a unit the $\frac{1}{3.5.12}$ of the entire work.

Then each man does 5 of these units each hour, and each woman does 3 of them.

$\therefore$ Ans. $6\frac{3}{7}$ hours.

5. Take as your unit $\frac{1}{2.3.4.5}$ of what the 6 men or 8 women or 10 children eat each day.

Then each man eats 20 units each day.
" woman " 15 "
" child " 12 "

$$\text{Hence the ans. } \frac{20 \times 100 \times 300}{1500 \times 1200};$$

$$\text{or } \frac{600000}{2700}, \text{ or } 222\frac{2}{9} \text{ days.}$$

6. A could finish it in 40 days, and B in 120.

7. Take as your unit $\frac{1}{3}$ of the quantity A drinks each day.

Then A drinks 3 units.
C " 6 units.
A and B " 8 units;
$\therefore$ B " 5 units;
$\therefore$ the cask contains $(3+6+5)$ 15 units;

$$\therefore \text{ A would drink it in } \frac{14 \times 15}{3}, \text{ or } 70 \text{ days.}$$

8, 9, 10. See 7.

11. Take as our unit the 7th of the one part, which is = the

8th of the other. Then the one part contains 7 units, the other 8 units; and these 15 units = 60;

$\therefore$ a unit = 4, or 28 and 32 are the divisions.

12. For each horse he pays £17, 10s.; for each sheep, £1, 9s. 2d.; for each pig, £1, 15s.; for each fowl, 3s. 6d.; therefore he pays altogether, £345.

13. £1256, 15s. $8\frac{4}{7}$d.

14. 96 days.

15, 16. See 14.

17. 5 hours a day.

18. See 17.

19. The boys under 10 receive £16, 13s. 4d.; therefore there are 300 boys.

20. £450.

21. 108.

22. £2, 15s. $6\frac{2}{3}$d.

23. £1.

24. 15 days.

25. 25 men.

26. 40 days.

27. £21, 14s. 11d.

28. See 27.

29. £6, 5s. $2\frac{2}{3}$d.

30. $16\frac{4}{5}$d.

31, 32, 33. See 20.

34. £280.

35. $\frac{3}{80}$.

36. Take as your unit $\dfrac{1}{3 \times 9 \times 4}$ of B's share, which is $=$ £12, 12s. $8\frac{7}{27}$d.; and A will have £202, 3s. $0\frac{4}{27}$d.; B, £1364, 10s. 4d.; and C, £1503, 9s. $10\frac{23}{27}$d.

37, 38, 39. See 36.

40. The value of each unit is £0, 12s. 4d., and he receives £407, 12s. 4d.

41. £4, 2s. 8d.

42. See 41.

43. See 41.

44. 5s. $1\frac{1}{4}$d.

45. See 44.
46. $9\frac{3}{4}$ days.
47. £341, 5s.
48. See 46.
49. $3\frac{3}{5}$ miles an hour.
50. 6 miles an hour.

CHAPTER VII.

1. £5, 4s. 6d.; ·25.
2. £1, 9s. $\frac{1}{3}$.
3. £13, 17s. 6d.
4. £76, 16s.
5. 4; ·06.
6. ·080025; 24 cub. ft. 810 in.
7. £2, 16s. 11d.; $22\frac{23}{30}$.
8. $9\frac{116}{121}$; $\frac{7}{208}$.
9. $\frac{8}{135}$.
10. £4, 0s. 9d.
11. 8757.
12. $2\frac{24}{317}$.
13. 17s. $6\frac{1}{2}$d. $\cdot 017541\dot{6}$.
14. £1. ·1390625.
15. 11s. $1\frac{1}{2}$d. ·11125.
16. 3 ro. 9 po. 29 yds. 11 ft. 36 in.: $\frac{1}{5}$d.
17. $\frac{1}{22}$.
18. Reduce them to fractions, whose numerators are 1.

Thus:— $$\frac{11 \times 4}{5 \times 9} = \frac{1}{1\frac{1}{44}}; \quad \frac{12 \times 3}{4 \times 10} = \frac{9}{10} = \frac{1}{1\frac{1}{9}};$$

$$\frac{10 \times 5}{6 \times 8} = \frac{25}{24} = \frac{1}{\frac{24}{25}}; \quad \frac{11 + 4}{5 + 9} = \frac{15}{14} = \frac{1}{\frac{14}{15}};$$

$\therefore \dfrac{11+4}{5+9}$ is the largest, next $\dfrac{10 \times 5}{6 \times 8}$, next $\dfrac{11 \times 4}{5 \times 9}$,

and $\dfrac{12 \times 3}{4 \times 10}$ the least.

19. 14s. $7\frac{1}{2}$d. ·14625.
20. $\frac{4}{9}$.
21. 755.
22. ·00875.
23. $\frac{1}{3}$ of a quart.
24. 9s. $1\frac{33}{35}$d. : $\frac{481}{350}$.
25. $\frac{3}{128}$ or ·0234375.
26. £1.
27. £3, 6s. a quarter.
28. £16, 2s. $7\frac{17}{40}$d.
29. 19s. $11\frac{1}{2}$d.
30. $1\frac{1}{5}$.
31. Gold will be $\frac{19}{38}$ of the silver.
32. $\frac{310}{511}$.
33. $1\frac{1}{72000}$ seconds.
34. ·15625.
35. $\frac{27}{220}$. 594 sq. yds.
36. ·015625 ; ·00704032 ; ·45058048 ; $\frac{176008}{390625}$.
37. 4s. 8d.
38. 3 lbs. 7 oz. 15 dwts.
39. ·00015625.
40. £30, 14s. $8\frac{2}{5}$d.
41. $3\frac{1}{2}$ hours.
42. 5s. 10d.
43. 2 cwt. 0 qrs. 26 lbs.
44. £28, 12s. $3\frac{3}{4}$d.
45. £611, 3s. 4d.
46. 30 of each.
47. £3, 9s. $6\frac{1}{4}$d. ·6952083̇.
48. 223·358, etc. 20·0574 oz.
49. 5s. $1\frac{1}{4}$d.
50. 8s. 2d.

CHAPTER VIII.

1. £8192 ; £4778, 13s. 4d.
2. £970, 17s. $5\frac{73}{103}$d.
3. £184, 7s. 8d.

4. 5 to 3 on him.
5. A ought to give C 640 points.
6. C has $\frac{9}{8}$ of B's.
7. 9 to 14 on or 14 to 9 against.
8. 17 to 13 on the 3rd, or 13 to 17 against.
9. One fourth.
10. 56.
11. C has $\frac{3}{5}$ of A.
12. C has $\frac{10}{33}$ of A's.
13. 5300 for B, 5220 for A.
14. $2\frac{1}{2}$ min.
15. 11 days.

16. If the 12 men do $\frac{4}{5}$ of work in 8×20 hours, they will complete the work in $\frac{5}{4}$ of (8×20) or 200 hours, and 1 man in 2400 hours.

The second lot of men have $\frac{7}{8}$ of 2400 or 2100 days' work, and to do this 15 men work 10 hours a day. Hence the answer $\dfrac{2100}{15 \times 10} = 14$ days.

17. Since 24 of A's hours' work and 27 of B's do $\frac{1067}{2261}$ of the work to be done, and 27 of A's hours' work and 30 of B's do the remaining $\frac{1194}{2261}$,

To equate A's hours' work in these two expressions,

(1) 24×9 of A's hours' work and 27×9 of B's do. $\frac{1067}{2261} \times 9$.
(2) 27×8 " 30×8 " $\frac{1194}{2261} \times 8$;
$\therefore$ 3 of B's hours will do $\frac{51}{2261}$ of the work;
$\therefore$ 1 of B's hours will do $\frac{17}{2261}$ "

Similarly, we can find 1 of A's hours' work will do $\dfrac{25\frac{1}{3}}{2261}$ of the work.

18. $2\frac{1}{2}$ miles.
19. 16 days.
20. 100 hours.
21. £15.
22. 880.

23. The 10 town workmen and 12 country ones will cost £4, 10s. more than the 15 town workmen.

24. 2 men.

25. A has $1\frac{1}{9}$, B $1\frac{4}{9}$, C $\frac{4}{9}$ respectively of the original sum.
26. Half.
27. The first kind. Saving on 100 inches of candle, $1\frac{303}{697}$d.
28. 2s. $2\frac{1}{2}$d.
29. £50.
30. ·14375.
31. £25.
32. A could complete it in 168 days, B 84, C 42.
33. 60 sheep.
34. £133, 6s. 8d.
35, 36. See 34.
37. There are 12 gallons more wine than water in cask A. Hence we must find a fraction equal to $\frac{2}{3}$ of which the terms differ by 12, which is $\frac{24}{36}$; ∴ there are 24 gallons of wine and 36 gallons of water in cask B; that is, 60 gallons in all.
38. Take $\frac{1}{9}$ of the pudding as your unit. Ans., 3d., 9d., 1s. 4d.
39. Remember that the difference in time between losing and gaining is found by adding together the loss and gain. Ans. 2 seconds.
40. £781, 1s. 8d. £2343, 5s.
41. £300.
42. £233, 6s. 8d.
43. 144.
44. 2 hours 24 minutes.
45. Thaler, 2s. 3d., and dollar, 4s. 2d.
46. £4, 7s. 11d.
48. 4s.
49. $2\frac{1}{2}$ miles.
50. £5, 19s. $5\frac{3}{5}$d.

CHAPTER IX.

1. $\cdot\dot{3}\dot{4}$ means $\frac{34}{100}+\frac{34}{10000}+\frac{34}{1000000}$, etc., *ad infinitum*.
2. $\frac{133}{99}$.
3. $\frac{114}{99}\times\frac{10}{57}=\frac{20}{99}=\cdot\dot{2}\dot{0}$
4. $\cdot 25\dot{9}$.
5. $\cdot 00\dot{6}=\frac{6}{900}$, or $\frac{2}{300}$.

6. 5.

7. $\cdot\dot{3}\dot{4}$ (septenary) $= \dfrac{34}{49} + \dfrac{34}{49 \times 49} + \dfrac{34}{49 \times 49 \times 49}$, *ad infinitum.*

8. $\frac{34}{66}$, a fraction in its lowest terms.

9. $\dfrac{4712 - 47}{770} = \frac{4643}{770}$, a fraction in its lowest terms.

10. $\frac{52}{8} \times \frac{80}{21} = \frac{520}{21} = 24\cdot\dot{2}3275180\dot{4}$.

11. $3\cdot1\dot{7}$ of £2, 6s. 3d. $= \frac{286}{90}$ of £$\frac{37}{16}$ = £7, 6s. $11\frac{2}{3}$d.

12. £11, 18s. 4d.

13. $\frac{2655}{700}$ = octonary $= \frac{1453}{448} = 3\cdot243303\dot{5}71428$.

14. It only refers to the point used in the ordinary or denary scale, whereas unit point is applicable to all scales.

15. Multiplied the number by $7 \times 7 \times 7$ or 343.

16. Divided the number by $8 \times 8 \times 8$ or 512.

17. £$2\cdot31\dot{4}$.

18. £3, 1s. $5\cdot4021$d. (quinary). Multiply by 40 (quinary) to reduce to shillings, and by 22 to reduce to pence.

19. £$0\cdot01458\dot{3}$.

20. £243, 2s. $0\frac{3}{10}$d.

21. £337, 9s. $2\frac{26}{125}$d.

22. £166, 0s. $6\cdot304512$d.

23. Any fraction whose denominator (fraction being in lowest terms) contains no factor but 2 and 3, as $\frac{1}{27}$ (denary), that is, $\frac{1}{23}$ duodenary, $\frac{1}{64}$ (denary), that is, $\frac{1}{54}$ (duodenary) or $\frac{1}{54}$ (denary), that is, $\frac{1}{46}$ duodenary.

24. $\dfrac{57}{99} + \left(\dfrac{2}{9} \times \dfrac{23}{10}\right) = \dfrac{285 + 253}{9.11.5} = \dfrac{538}{9.11.5}$.

25. 29 times and 12 forty-fifths or 4 fifteenths over. If we had been asked for the number of times and fraction of a time, the answer would have been $29\frac{12}{17}$. For further discussion on this subject, see Chapter X.

26. 3 tons 4 cwts. 2 qrs. 23 lbs. 5 oz. 14 drs.

27. A sum of money, like 5s. $7\frac{1}{2}$d., of which the farthings are divisible by 3 (on account of the 3 in the 12 used in reducing the pence to shillings) and 5 (on account of the 5 in the 20 used in reducing the shillings to pounds), and also of

which the shillings are divisible by 5, on account of the 5 in the 20 as before.

28. The selling price $\frac{93}{90}$ of 1000 or $\frac{9300}{9}$;

$\therefore$ my profit is £33, 6s. 8d.

29. My buying price is £877, 15s. $6\frac{2}{3}$d;

$\therefore$ my profits are £122, 4s. $5\frac{1}{3}$d.

30. $\cdot 8\dot{5} = \frac{76}{80}$ nonary $= \frac{69}{72}$ or $\frac{23}{24}$ denary. Ans. £30, 7s. 6d. But since 729 denary = 1000 nonary, £729 × $\cdot 8\dot{5}$ = £855$\cdot\dot{5}$ (nonary), which is = £698, 12s. 6d., and this gives, as before, £30, 7s. 6d.

31. £18, 15s.

32. $\frac{2}{9}$ of my receipts are 1000;

$\therefore$ my receipts are 4500.

33. See 32.

34. £371, 5s. 0d.

35. £100700.

36. $\cdot 00125$.

37. £165, 12s.

38. C's share, £53, 15s. $10\frac{10}{21}$d.

39. $36\cdot\dot{3}\dot{6}$.

40. $66\cdot 2\dot{5}\dot{4}$ times.

41. $29\cdot 25$.

42. 1400.

43. $5\cdot 26246545$, etc.

44. £5, 15s. $8\cdot 376$d. (nonary); note the 15s. means 14s. in the denary.

45. $\cdot 7$.

46. $\dfrac{36994}{ee0} = \dfrac{t854}{2e9}$ (duodenary).

47. $\frac{2102}{50}$ (sen.) $= \frac{470}{30}$ (den.) $= \frac{47}{3}$.

48. $\cdot 00001304313$, etc., lbs.

49. $\cdot 000026e3$, etc., acres.

50. $\dfrac{18\cdot 7 \times e\,.\,7 \times 15\,.\,e \text{ cubic feet}}{22\,.\,73 \text{ square feet}} = 114\cdot 69$, etc., $160\cdot 51$, etc.

CHAPTER X.

1. $2\frac{1}{12}$ lbs. copper.

2. Multiply the first quantity by 4 and the second by 3, and the difference will give you the price of the tea.

Ans. Tea 30d., sugar 8d.

3. If A and B together earn 40s. in 6 days, A and B earn $\frac{40}{6}$, or 6s. 8d. in 1 day.

Similarly A and C earn $\frac{54}{9}$, or 6s. in 1 day;

$\therefore$ B earns 8d. more than C,
or B's less C's earnings are 8d.

But B's and C's earnings are $\frac{80}{15}$, or 5s. 4d. each day;

$\therefore$ B's earnings are $\frac{1}{2}$ of $(5-4+8\text{d.})$, or 3s. a day.

C's 2s. 4d., and A's 3s. 8d.

4. Add on the half guinea the difference between the guinea and a half and a guinea, and you get easily 4 sovereigns, 59 shillings, and 55 sixpences.

5. A prime number is one that has no factors but itself and unity.

2.2.2.2.2.2.3.13.53.

6. A ran 50 yards more than B, which is $\frac{1}{3}$ of the course or 150 yards.

7. $\frac{1}{273}$, $\frac{5}{182}$, $\frac{1}{42}$, $\frac{2}{15}$, $2\frac{5}{92}$.

8. The significant figures being the same in the dividend and divisor, they will be the same in the quotient. (1) ·0009027. (2) 9·027. (3) 1.

9. Since every common measure of two numbers must be a measure of the G. C. M., the G. C. M. must be a common multiple of all the measures; and if it be not the least, let there be some number less than the G. C. M., which is their common multiple; but the G. C. M. itself is a measure of the G. C. M., hence a less number is a multiple of a greater, which is impossible.

10. 24855 mls. 2 fur. 14 po. 5 yds. 8 in.

11. He loses 25 francs or £1.

12. 2·2 lbs. or 2 lbs. $3\frac{1}{5}$ oz.

13. If I purchase them all at the rate of 3 for 2d., I spend $80\frac{1}{3}$ pence too much. The difference between $\frac{2}{3}$ and $\frac{1}{2}$ is $\frac{1}{6}$, but the difference is double if taken from one lot and put into the other. Since $80\frac{1}{3}$ contains $\frac{1}{3}$d. 241 times, I must remove the eggs to be purchased for 241 pence into the 2 a penny basket, leaving $659 - 482$ or 177 in the 3 for 2d. basket.

14. 1·093633 yds. nearly.

15. 2s. 6d.

16. Since to produce 9 from multiplying 4, I must multiply 4 by $\frac{9}{4}$. Any number of lbs. of tea therefore multiplied by $\frac{9}{4}$ will give me the price in shillings, *e.g.* 8 lbs. cost 18s., 26 lbs. cost 58s. 6d.

17. 61·162984.

18. £1125, 10s. $2\frac{143}{1250}$d.

19. If 8 gold coins + 9 silver coins = 6 gold coins + 19 silver coins,

$$2 \text{ gold coins} = 10 \text{ silver ones.}$$
$$1 \text{ gold coin} = 5 \text{ silver ones.}$$

20. 56 miles.

21. 10 miles.

22. $\frac{7}{12}$ of a mile.

23. Daughter, £500; son, £1000; widow, £4000.

Subtract the £500 for the widow, and divide remainder by $(3 + 4 + 7)$, which will give a daughter's share.

24. 19 days.

25. Ans. 80. See par. 8.

26. $4\frac{1}{2}$ hours with the stream, and $7\frac{1}{2}$ hours against the stream.

27. 14s., 21s., $52\frac{1}{2}$s.

28. 45 gallons.

29. 20 miles.

30. 60 eggs.

31. £144.

32. Take as your unit the $\frac{1}{2.4.5.9}$ of the daily wage. Thus A receives 45 units each hour, B 40 units; and C 36 units.

Ans. A 15s. $11\frac{1}{4}$d.; B 15s. 10d.; C 15s. 9d.

33. 225 and 125.

34. $\frac{2}{75} = \frac{\frac{8}{3}}{100}$; $\frac{\frac{5}{16}}{100}$.

35. £4, 3s. 5d.

36, 37. See 35.

38. Since the elements $\frac{75}{364}$ and $4\frac{1}{3}$ are introduced twice, without a knowledge of the properties of numbers in proportion, it would be very difficult, if not impossible, to find them.

39. 17s. $11\frac{1}{4}$d.

40. $\frac{19}{2749}$. $\frac{693500}{173187}$ or $4\frac{752}{173187}$.

41. Adding $\frac{141}{2277}$ of a penny.

42. 65 lbs.

43. $\frac{7}{10}$.

44. B gets twice as much feed from the field as A; ∴ B pays £40 and A £20.

45. B catches A in 12 hours, and has travelled 20 myriametres.

46. Take $\frac{1}{2.3.5}$ of what A or B can do in 6 days or C can do in 10 days as your unit. C would finish it in 10 days.

47. £3, 4s.

48. Half a quartern loaf.

49. They are diminished by $\frac{1}{3}$ of their wages.

50. Take $\frac{1}{3.4.8\frac{1}{4}}$ of the work as your unit.

Ans. $4\frac{1}{2}$ hours.

Miscellaneous Answers.

I.

1. 379.

2. 175 with a remainder 28.

3. 374 (say) taken (say) 3 times is 374 + 374 + 374.

4. 6206569.

5. 76.

6. 396.

7. £31, 15s. 3d.

8. A, 2 min. 40 sec.; B, 2 min. 56 sec.; and C, 3 min. 40 sec.

II.

9. $751310\frac{1}{5}$.

10. 144 is half as large again as 96, and a third as large again as 108, which two numbers differ by 12.

11. 56.

12. 126.

13. 37.

14. A has £140, 14s. 3d.; B £120, 14s. 3d.; C £117, 14s. 3d.

15. $\dfrac{4\frac{1}{4}}{5\frac{9}{56}}$.

16. $11\frac{1}{10}$.

III.

17. $2\frac{33}{40}$.

18. 40.

19. $\dfrac{1}{1\frac{2}{3}}$; $\dfrac{1}{1\frac{7}{8}}$, $\dfrac{1}{1\frac{13}{20}}$,

of which $\dfrac{4}{7\frac{1}{2}}$ is the least, since $1\frac{7}{8}$ is the greatest of the denominators.

20. $\frac{11}{63}$.

21. $\frac{847}{36}$.

22. A had 18s.; B £2, 2s.; and C £5, 15s. 6d.

23. 48.

24. 490.

IV.

25. 12 times and $\frac{3}{28}$ over.

26. 1 mile 4 furlongs.

27. 50.

28. £534, 3s. 3d.

29. 224068.

30. 41.

31. The difference between $\frac{1}{4}$ and $\frac{1}{5}$ is greater than between $\frac{1}{5}$ and $\frac{1}{6}$; hence his losses on the 4 a penny apples are greater than his gains on the 6 a penny ones.

32. Since he neither loses nor gains on the apples he buys for 5 a penny, it makes no difference how many he buys of them; of the others he must buy two of the 4 a penny to three of the 6 a penny.

V.

33. 245.
34. 29 times and 1 third of a dr. over.
35. 2668140.
36. $\frac{23}{42}$.
37. No more times; but there is a remainder 2 units in the former case.
38. 8s. $8\frac{52}{385}$d. $\frac{3341}{15785}$.
39. 1946.
40. 30.

VI.

41. 100172.
42. 11720.
43. One-sixth of the length of the stake.
44. 142 times and 14 thousandths over.
45. $953\frac{11}{16}$.
46. He goes $553\frac{11}{16}$ yards in a minute.
47. 3360.
48. $\frac{68}{157}$; the other fractions are $\frac{1}{2}$, $\frac{3}{7}$, $\frac{13}{30}$, and these four fractions reduced to their common denominator are

$$\frac{14280,\ 16485,\ 14130,\ 14287}{157 \times 2 \times 7 \times 15},$$

whence it is seen that $\frac{1}{2}$ is greater, $\frac{3}{7}$ less, and $\frac{13}{30}$ again greater than the fraction.

VII.

49. 14s. $7\frac{1}{2}$d. $\frac{39}{62}$.
50. £10, 10s.
51. 45 times and 35 hundredths over.
52. $\cdot 0\dot{7}1428\dot{5}$.
53. 4 guineas.
54. 2·106.
55. Any sum of money exactly divisible by 9, since it has to

be divided by 3.4.20.3, and unless the 3×3 appear in the sum as factors the decimal will not terminate.

56. $\frac{1}{1\frac{5}{12}}$; $\frac{1}{1\frac{13}{35}}$; $\frac{1}{1\frac{3}{7}}$, whence the third is the least and the first the greatest; and the middle one is $\frac{37}{840}$ greater than the least.

VIII.

57. $\frac{6}{25}$.

58. £3, 2s. 8d.

59. 104 times and 71 fourteenths of a sixteenth of a grain over.

60. 123·442. The ·442 means $\frac{4}{5}+\frac{4}{25}+\frac{2}{125}$.

61. 11 times and $\frac{1012}{1000}$ all expressed in quaternary, or 5 and $\frac{70}{64}$ over if expressed in denary.

62. 40 gallons.

63. $\frac{11 \text{ gallons.}}{16\frac{1}{2} \text{ gallons.}}$

64. $\frac{177}{4}$.

IX.

65. 300; 300; ·00003, 3000.

66. 13·68.

67. 691200.

68. 100.

69. The first quantities all being capable of reduction to the same denomination, I get these results in money, viz. (1) $\frac{4\text{s.}}{£5} \times 6\text{s. } 8\text{d.}$, or $3\frac{1}{5}$d.; (2) $\frac{£5}{4\text{s.}} \times 6\text{s. } 8\text{d.}$, or £8, 6s. 8d.; (3) $\frac{4\text{s.}}{6\text{s. } 8\text{d.}}$ of £5, or £3.

But in the other case I can only have two results, viz. (1) $\frac{4 \text{ lbs.}}{3 \text{ cwts.}}$ of 10 miles, or $\frac{5}{42}$ mile; and (2) $\frac{3 \text{ cwts.}}{4 \text{ lbs.}}$ of 10 miles or 840 miles.

70. B rides 2 miles in $\frac{2}{9}$ of a hour or 9 miles an hour.

71. 30 miles.

72. $51\frac{81}{89}$ days.

X.

73. 314.
74. 30·12.
75. £680, 2s. $3\frac{1}{8}$d.
76. 5 shillings.
77. No remainder after dividing by 7; but 9 after dividing by 12.
78. One man's work is equivalent to three boys'.
79. 294.
80. 3·642782, etc.

XI.

81. 11.
82. £367, 10s.
83. £18, 10s. $0\frac{120}{147}$d.
84. Multiply by $\frac{10000}{16}$, which gives the answer 232000.
85. Split up the 12181 into 12100, 70, and 11; and to get the 12100 multiply first line by 11.
Ans. 574005263.
86. $\frac{1}{6}$.
87. 6 and 12.
88. The younger daughters, £812, 10s. each; the eldest daughter and 2 younger sons, £1312, 10s. each; the eldest son, £2312, 10s.; and the widow, £10500.

XII.

89. £$\frac{49}{107}$.
90. 459.
91. $\frac{77 \times 29}{60}$ gallons.
92. 2.
93. There must be either no, or an even number of, odd digits.
94. Since $(2+4)-6=0$, it will divide without remainder by (radix + 1) or 8; and since $2+4+6$ is divisible by one less than the radix or 6, the number will divide by 6 without remainder.
95. 55.
96. £1.

PART II.

CHAPTER XI.

1. A ratio is the statement of two quantities or numbers to be compared as to how many times the first contains the second. Whereas the measure of a ratio is the abstract number which shows how often the first contains the second; *e.g.* the measure of the ratio £8 : £2 is 4; the £8 : £2 is = 4. You could not say that the ratio of £8 : £2 is the same as 4.

2. You compound the measures of the ratios.

3. $\frac{3}{5} - \frac{4}{7} = \frac{21-20}{5 \times 7}$, hence the former is larger by $\frac{1}{35}$ than the second.

4. 4s. : 5d. = $\frac{48}{5}$ and 3 oz. : 12 lbs. = $\frac{3}{192}$, and these compounded = $\frac{48}{5} \times \frac{3}{192} = \frac{3}{20}$.

5. Yes, because they can both be reduced to grains, hence 1 oz. avoirdupois : 1 oz. troy = $\frac{7000}{16}$: 480, the measure of which is $\frac{175}{192}$.

6. $3\frac{19}{24}$.

7. $4\frac{4}{9}$.

8. $\frac{9}{40}$.

9. $4\frac{4}{7}$.

10. In a direct proportion quantities are so connected together that any increase or decrease in the one is attended by a corresponding increase or decrease in the other, as men and wages, weight and price; whereas in inverse proportion quantities are so connected together that any increase or decrease in the one is attended by a corresponding decrease or increase in the other, as time and men.

11. $5\frac{1}{4}$.

12. $6 : 3 = 4 : 2$
$6 : 4 = 3 : 2$
$3 : 6 = 2 : 4$
$3 : 2 = 6 : 4,$
and these four backwards,

e.g. $2 : 4 = 3 : 6$.

13. $8\frac{3}{4}$.
14. $4\frac{2}{7}$.
15. $1\frac{3}{4}$.

16. $\dfrac{32 \times 3 \times 6}{55}$, or $10\dfrac{26}{55}$.

17. That the 3 men do in 10 weeks what the 5 men do in 6 weeks.

18. When the first term corresponds with the last and the second with the third.

19. $10\frac{1}{2}$.
20. $15 \cdot 8\dot{6}$.
21. $\cdot 7\dot{2}8571\dot{4}$.
22. $1 \cdot 9\dot{7}647058823529411\dot{i}$.
23. $7 \cdot 2\dot{3}$.
24. $10\frac{1}{2}$.
25. £1.
26. 1 oz. 2 drs.
27. 12 cwts. 1 qr.
28. 10s.
29. The latter is greater by $\frac{3}{5}$.
30. The latter is greater by $\frac{10}{63}$.
31. 41 men : 35 men.
32. 6 lbs. : 77 lbs.
33. £1, 12s. : £3, 5s.

34. $\dfrac{1\frac{1}{2}}{1\frac{1}{2}+2}$, or $\dfrac{1\frac{1}{2}}{3\frac{1}{2}}$.

35. $\frac{30}{27}$.

36. $\dfrac{\text{£}2,\ 10\text{s.}}{\text{£}44,\ 3\text{s. } 4\text{d.}} = \dfrac{3}{53} = \dfrac{6}{106}$.

37. $\dfrac{£41,\ 13s.\ 4d.}{£44,\ \ 3s.\ 4d.} = \dfrac{50}{53} = \dfrac{100}{106}$.

38. $\dfrac{\frac{2}{3}}{1}$.

39. $\frac{30}{20}$.

40. $17\frac{1}{3}$.

CHAPTER XII.

1. $213\frac{1}{5}$.
2. 4 lbs. 1 oz. 9 dwts. $22\frac{106}{125}$ grs.
3. 44.
4. £810, 12s. 6d.
5. £31, 5s.
6. 18.
7. 18 days.
8. 9d.
9. 162 bushels.
10. 2 days.
11. 15 yards.
12. £1, 10s.
13. 15 men.
14. £96, 17s. 6d.
15. 108.
16. 108 miles.
17. $56\frac{1}{4}$ miles.
18. £560.
19. 9.
20. £48.
21. £240,000,000.
22. £631, 10s.
23. Remember the difference between rowing with and against a stream at the rate of $2\frac{1}{2}$ miles an hour is 5 miles an hour.
24. £2500.
25. £3050, 15s. 6d.

26. 1s. $7\frac{1}{4}$d.
27. £3, 3s. 3d.
28. $4\frac{1}{2}$.
29. 12 minutes.
30. 18 ac. 3 ro.
31. 19.
32. 30.
33. 1s. $0\frac{1}{2}$d.
34. 13s. 4d.
35. $47\frac{4}{13}$ miles.
36. £33, 15s.
37. 1 yr. 3 mo.
38. 36 minutes.
39. $10\frac{2}{11}$ miles an hour.
40. 1320.
41. 10s.
42. $11\frac{1}{4}$d.
43. $3\frac{2}{21}$d.
44. 33 gallons.
45. $1\frac{1}{2}$ gallons.
46. 9s. $4\frac{6}{7}$d.
47. $7\frac{1}{5}$d.
48. 2 lbs. $14\frac{2}{3}$ oz.
49. 12s. a gallon.

50. Find the buying price by reciprocity. Ans. $\dfrac{101\frac{1}{19}}{100}$.

51. The price of mutton : that of salmon as $16\frac{1}{4}$: 20, or 13 : 16. The proportion is inverse.

52. 30 days.

53. This question becomes, 'Find a ratio in gallons whose terms differ by $3\frac{1}{2}$ which is equal to $\frac{16}{18}$.' Ans. 28 gallons.

54. Ans. 14s. for a bushel and a half. For 1 quart $3\frac{1}{2}$d.

55. 8 men : 5 men $= 6\frac{2}{5}$ days : 4 days. The number of men ought to be integers. This states what 8 men can do in 4 days 5 men would do in $6\frac{2}{5}$ days; of course there are many other statements which would satisfy the condition—

e.g. 16 men : 13 men $= 2\frac{6}{13}$ days : 2 days.

56. 6s. 8d.
57. £23, 8s.
58. £405, 7s. 6d.
59. $6\frac{21}{25}$d.
60. £6120.

CHAPTER XIII.

1. £5 for every £100, or 5 eggs for every 100 eggs, or 5 anything for 100 of those things.
2. £19, 14s. $1\frac{1}{2}$d.
3. £55, 10s.
4. 4s. 8d.
5. $7\frac{1}{4}$d.
6. 3s. 1d.
7. £5 gain.
8. Lost 5 per cent.
9. 25 per cent. on the price and carriage.
10. $27\frac{1}{2}$ per cent.
11. $2\frac{1}{12}$ per cent.
12. £631, 10s.
13. He loses 25 francs, or $1\frac{7}{13}$ per cent.
14. 7992 rupees.
15. $154\frac{11}{12}$.
16. $4\frac{1}{6}$ gained.
17. $48\frac{1}{2}$.
18. £78, 12s. $9\frac{3}{4}$d.
19. £88, 11s.
20. £24.
21. £147, 15s.
22. £560.
23. £2500.
24. £5, 1s. 3d., 5 per cent. loss.
25. £11, 2s. $2\frac{2}{3}$d.
26. £110, 5s.
27. $4\frac{28}{45}$ per cent., or 8d. a gallon.
28. $13\frac{859}{1587}$ per cent., or £4, 7s. 11d.
29. $\frac{1}{3}$ of a quart.

30. 1s. $1\frac{7}{8}$d.
31. 144.
32. The first kind. There is $1\frac{303}{697}$ saved on every 100 inches. Saving on £100 in using first £1, 6s. $8\frac{160}{187}$d.
33. 17s. 6d.
34. 4s. 8d.
35. £2, 12s.
36. £780; 13s.
37. 720.
38. $12\frac{1}{2}$ per cent.
39. £701, 11s. 5d. to widow; £694, 11s. 1d. to sister; £680, 10s. 6d. to niece. Net sums £670, £663, 6s., and £649, 18s.
40. £5166, 13s. 4d.; £143, 1s. $1\frac{1}{3}$d.
41. $7\frac{43}{51}$.
42. $\frac{6}{5}$.
43. 5 lbs. or any multiple of 5 lbs.
44. $\frac{1}{5}$ shilling or some multiple of it.
45. $\frac{9}{25}$ of a penny.
46, 47, 48, 49. See 45.
50. 6s. 8d.

CHAPTER XIV.

1. When the incomplete ratio depends on two or more ratios which have to be compounded together.

2. (1) If 9 men will earn some money in 20 days, in how many days will 6 men earn it? Ans. $\frac{9 \times 20}{6}$.

(2) If some men will earn £8 in $\frac{9 \times 20}{6}$ days, in how many will they earn £10. Ans. $\frac{9 \times 20 \times 10}{6 \times 8}$ or $37\frac{1}{2}$ days.

3. If 6 lbs. cost 12s., 2 is the constant multiplier which will make the weight expressed in pounds the price expressed in shillings.
4. 12.
5. 14.
6. £51, 4s. $3\frac{1}{2}$d.
7. 28 horses.
8. 11 days.
9. 21 days.
10. 5 hours.
11. 81 men.
12. 4 days.
13. £33, 1s. 6d.
14. £21, 7s. 6d.
15. 11 hours.
16. 15 men.
17. £7, 10s.
18. 6 days.
19. 9 days.
20. 21 days.
21. 8 days.
22. £50.
23. By the nature of the question, it will be seen that the boys do no work whatever, as it is easily shown that the same number of men, with different number of boys, do the same amount of work. This is a question taken from the Cambridge Local Examination Paper, and is instructive to a certain extent. If the boys are useless, the answer is 2. Since if 3 men can do half in 6 days, 6 would do $\frac{1}{6}$ in 1 day; hence, with the 4 men 2 must be put on.
24. £48, 16s. $4\frac{12}{13}$d.
25. 48 lines.
26. 90 navvies.
27. 16 hours.
28. 88 per cent.
29. 8 hours.
30. £53, 10s.
31. 8 days.
32. $11\frac{1}{13}$.

33. 63 : 20.
34. See 33.
35. 45 men.
36. $\frac{42}{5}$.
37. 6.
38. 15 : 9.
39. £18,600.
40. $41\frac{79}{81}$.
41. 2s.
42. £2800.
43. 500 roods.
44. 30 days and $\frac{2}{15}$ of 9 hrs. (another of their days).
45. 165 men.
46. 5 per cent.
47. **48.** See 46.
49. 855 girls pass; grant, 18s.

CHAPTER XV.

1. 8 per cent. per annum means that 8 units are paid for the use of 100 units for 1 year.

$\frac{1}{2}$d. per 2s. a month means in the same way that $\frac{1}{2}$d. is paid for the use of 2s. every month.

2. If I pay £4 for the use of £100 for 1 year, what shall I pay for the use of £500 for 4 years?

3. £5.
4. £500.
5. No difference.
6. £128, 12s. 6d.
7. 4 per cent.
8. £16, 11s. $11\frac{51}{128}$d.
9. See question 8.
10. £127, 9s. $4\frac{1}{2}$d.
11. £53, 2s. 6d.
12. $2\frac{1}{8}$.

13. £21, 17s. 6d.
14. £6250.
15. £475, 15s. 3d.
16. $2\frac{5}{8}$.
17. £3546, 4s. $8\frac{19}{40}$d.
18. £50,000.
19. £275.
20. 5d.
21. 225 years.
22. $3\frac{1}{3}$ per cent.
23. $12\frac{1}{2}$ per cent.
24. £662, 4s. $11\frac{11}{50}$d.
25. £7500.
26. 2390 years.
27. £500.
28. £100.
29. 2s. 6d. + 8s., or 10s. 6d.
30. $\frac{1}{20}$, or ·05, or call the pounds of the principal shillings.
31. If the unit be changed into shillings, the multiplier becomes unity, and the number of shillings in the interest is *the same* (the multiplier being unity) as the number of pounds in the principal.
32. $\frac{1}{12}$, which is a better form than $\cdot08\dot{3}$.
33. 1 year.
34. $3\frac{1}{3}$.
35. The interest on any principal at 1d. in the florin is $\frac{1}{24}$ of the principal, and this expressed in pounds is 10 times the principal. But the interest on 100 times this principal at 10 per cent. is also ten times the same principal; hence our theorem.
36. Here, if we take £100 as our principal, $\frac{100}{240}$ is the interest, and $\frac{1}{12}$ is the time; hence our answer is 5 per cent.
37. 20 per cent.
38. $38\frac{1}{48}$ per cent.
39. By fractional method to find the interest at $5\frac{1}{2}$ per cent. we multiply by $\frac{5\frac{1}{2}}{100} \times$ time, and $\frac{5\frac{1}{2}}{100} = \cdot055$.
40. £400.

41. $39\frac{13}{33}$ per cent.
42. £1604, 1s. 3d. 5 per cent.
43. £1475.
44. £900 at 5 per cent., and £300 at 4 per cent.
45. £50 and £45.
46. £96, 17s. 9d.
47. £16537·7575, etc.
48. 4 years.
49. £$95\frac{205}{1261}$.
50. What is the difference of rate, if the interest on £$95\frac{205}{1261}$ for 15 years is the same as that on £500 in 3 years at 5 per cent.?

CHAPTER XVI.

1. Interest is calculated on a present value, and discount on a future value of a sum of money.
2. £100.
3. It is greater by the interest on the difference between the present and future values of the sum; that is, the discount.
4. £2, 10s.
5. £80.
6. $4\frac{6}{61}$.
7. £757, 10s.
8. £4511, 2s. $2\frac{2}{3}$d.
9. See answer to 3.
10. If £97 amounts in some time at some rate to £100, since 97 is $\frac{97}{100}$ of 100; ∴ if the present worth is $\frac{97}{100}$ of the future value, some multiples of 97 and 100 will represent the present and future values of the sum of money, and by reciprocity the future value will be $\frac{100}{97}$ of the present value.
11. 3 per cent.
12. $\frac{1}{20}$.
13. $\frac{1}{11}$.
14. £$\frac{1000}{51}$, or £19, 12s. $1\frac{15}{17}$d.
15. 5 years.
16. The difference between the interest and the discount being

the interest on the discount, we at once can find the rate 3 per cent., which enables us to find the principal £84.

17. £570.

18. The future value is greater by the present worth than the interest on the present worth, which is the amount.

19. £260.

20. £300.

21. £270, 16s. 8d.

22. See 21.

23. £3, 5s. 1d.

24. See 23.

25. £3, 5s. $8\frac{81}{100}$d.

26. Both are £8, 2s. 6d.

27. £6060.

28. £49, 10s.

29. See 28.

30. £745, 0s. 5d.

31. To find the interest and see if the difference between its discount is the interest on the discount.

32. £963, 19s. $10\frac{254}{287}$d.

33. $\frac{2480287}{3320000}$ of a year.

34. Each equal to £10, 10s.

35. The former sum is the present value of the latter.

36. £63, 5s. 9d.

37. £5.

38. First find $4\frac{1}{2}$ per cent., and then the present value is easily found, £787, 8s. $0\frac{48}{127}$d.

39. £23, 2s. $4\frac{1}{2}$d.

40. Present value, £816, 17s. 3d.

41. £29, 3s. 4d.

42. 18s. $2\frac{2}{11}$d.

43.

44. $\frac{1}{21}$.

45. $\frac{200}{203}$.

46. $\frac{106}{100}$, or $\frac{53}{50}$.

47. £2000.

48. 18s. $2\frac{1174}{7389}$d.

49. £46, 15s. 0d.

50. $\frac{1}{401}$. No.

CHAPTER XVII.

1. $\frac{9}{24}$.
2. 18 : 30.
3. 60 : 40.
4. It is the same as dividing 100 into 2 parts in the ratio of 3 : 2.

$$\text{Since } \tfrac{1}{2} : \tfrac{1}{3} = 3 : 2.$$

5. 98 : 105.
6. 80 lbs. at 5s. and 100 lbs. at 4s.
7. £48, 15s. and £56, 5s.
8. $9\frac{1}{4}$ and $6\frac{1}{4}$.
9. £9, 4s. $7\frac{5}{13}$d.; 15s. $4\frac{8}{13}$d.

10. Multiply the original number by one of the multipliers and add it to the known difference; then this sum, divided by the sum of the multipliers, will give the number to be multiplied by the other multiplier; thus: $\dfrac{8+10\times 2}{5+2}=4$, the number to be multiplied by 5; hence the parts are 4 and 6, and $4\times 5-6\times 2=8$.

If we multiply the 10 by the 5 and add it to 8, and divide result by 7, we get as our two parts $8\frac{2}{7}$ and $1\frac{5}{7}$, which will also satisfy the conditions; thus—

$$8\tfrac{2}{7}\times 2-1\tfrac{5}{7}\times 5=8.$$

11. If I add to the difference between one part taken five times and the other part taken twice the whole number taken twice, I have the former part taken five times and also taken twice; or the former part taken seven times; hence I can find this former part by dividing the sum of the known difference and twice the original number.

12. 11 and 14, or $14\frac{4}{7}$ and $10\frac{3}{7}$.

13. 110 and 112.

Since the second number is 2 greater than the first, if I add to the difference between the two products 4 times 2, I know at once that 5 times less 4 times the smaller number is 102 + 8, or 110; and therefore the greater number is 112.

14. If the difference between the original numbers

multiplied by the larger multiplier is greater than the difference of the multiples, I must multiply the smaller of the two numbers by the larger multiplier.

15. No; since 5×7 is greater than 4.

16. Anything greater than 35, not necessarily as great as 36.

17. $9\frac{1}{3}$ and $6\frac{2}{3}$.

18. The sum of the multiples must be greater than the product of the original number and the lesser multiple, and less than the product of the original number and the greater product.

19. No.

20. 6 and 4. This is not difficult if you remember that the sum of the sum and difference of two numbers equals twice the greater number.

21. Twice as many pounds of the dearer than the cheaper.

22. 60 lbs. at 2s., 90 lbs at 2s. 2d.

23. Three gallons of the cheap to one of the dear.

24. 1 gallon.

25. The buying price of each pound is $\frac{90}{100}$ of $\frac{85}{100}$ of $\frac{25}{1} + \frac{10}{100}$ pence; and the profit is $14\frac{384}{769}$ per cent.

26. 8 feet.

27. 20 lbs.

28. $8\frac{1}{8}$ lbs.

29. 30 lbs.

30. 70 lbs and 30 lbs.

31. Take moments round F.—

$$101 \times 1 + 2 \times 2 + 4 \times 3 + 8 \times 5 = 2 \times \text{force required};$$
$$\therefore \text{force} = 78\tfrac{1}{2} \text{ lbs.}$$

32. $193\frac{1}{2}$.

33. $38\frac{1}{2}$.

34. $2\frac{2}{5}$ ft. from A.

35. $\frac{3}{4}$ of a foot from middle towards the end where the 10 lbs. is.

36. 20 lbs.

37. $21\frac{4}{5}$ feet.

38. Third kind where the power is between the weight and the fulcrum.

39. $5\frac{5}{9}$, $4\frac{4}{9}$.

40. 9, 8.

41. $9\frac{4}{13}$, $7\frac{9}{13}$.
42. $8\frac{10}{13}$, $8\frac{3}{13}$.
43. $\frac{1}{25}$ mixture : $\frac{1}{30}$ mixture $= 9 : 5$.
44. A, £292, 16s.; B, £24, 8s.
45. Boys have £100 and the girls £150.
46. Each boy has £$\frac{100}{60}$, and each girl £$\frac{150}{40}$;
$\therefore$ a boy's share : girl's share $= 4 : 9$.
47. 336 lbs.
48. The true lb. seems to weigh $\frac{99}{100}$ lbs. if it is placed in the pan attached to the longer arm, and $\frac{100}{99}$ if in the other pan.
49. $1\frac{2}{3}$ ft. from A.
50. Since the pressure on the fulcrum $= 107$ lbs. Taking moments round A, we find that clockwise we have a moment represented by 1106 to counteract by an upward pressure of 107 lbs.; this, therefore, must be placed $10\frac{36}{107}$ ft. from A.

CHAPTER XVIII.

1. 1·06.
2. £280, 18s.
3. £30, 18s.
4. £16, 8s. $10\frac{8}{25}$d.
5. £174, 10s. $1\frac{2808}{3125}$d.
6. £1511, 12s. 1d.
7. £1397, 11s. 3d.
8. £614, 2s. 6d.
9. Divide the difference by the principal.
10. £2852, 18s. $3\frac{27}{40}$d.
11. Practically this is just the same as if the money were due 3 years hence at $4\frac{1}{2}$ per cent. calculated annually. Ans. £2500.
12. $P \times 1{\cdot}04$ is the amount at end of first year, and $P \times 1{\cdot}04 \times 1{\cdot}035$ is the amount at the end of the second year; hence the principal will be found to be £650.
13. £190, 0s. $1\frac{1}{2}$d. very nearly.
14. £50, 12s. 6d.

15. £1458, 12s. $1\frac{4}{5}$d.
16. £484, 11s. $10\frac{1}{2}$d.
17. £2, 8s. $8\frac{1}{4}$d.
18. This difference being only that on two years; and since the compound interest for one year is the same as the simple interest for the same time, the difference, £2, 8s. $11\frac{1}{4}$d., is the interest on the interest for one year; hence, if we know the principal we can find the rate, or if we know the rate we can find the principal.
19. £104.
20. Since to find the amount I multiply the principal by 1·05 and 1·0375 to obtain the amount, I can find the 1·0375 by dividing the A by P × 1·05, all three of which I know.
21. $\frac{7}{8}$ of a year.
22. See 21.
23. £4802, 15s. $3\frac{3}{4}$d.
24. S. I. £4, 6s. 0d.; C. I. £4, 12s. $8\frac{49}{4000}$d.
25. £391, 13s. $7\frac{1}{5}$.
26. S. I. £1, 11s. 6d; C. I. £1, 12s. 4·4534d.
27. £128, 3s. $2\frac{3}{4}$d.
28. 11s. $6\frac{2862}{3125}$d.
29.
30. £305, 17s. 9·570765d.
31. £11, 12s. 7d.
32. £881, 4s. $10\frac{7}{80}$d.
33. £1500.
34. Divide the amount by P × 1·03 × 1·01.
35. £1, 1s. $9\frac{36279}{2560000}$d.
36. £103, 11s. 0·0576d.
37. See answer to 12.
38. £40.
39. Each carat of a 2 carat diamond is double the value of the carat in a single carat diamond; hence the large diamond is worth 2 × 2 the smaller, or 2^2. The value of a 200 carat diamond is £$\frac{200 \times 200}{6 \times 6}$ of 9 = £10,000.
40. £27.
41. 25 : 64.

42. $\frac{64}{25}$ of 20, or $51\frac{1}{5}$ sq. ft.
43. $8^3 : 11^3$, or $512 : 1331$.
44. $\frac{512}{1331}$ of 363, or $\frac{1536}{11}$, or $139\frac{7}{11}$.
45. 9 : 16.
46. $\frac{16}{9}$ of 27, or 48 super. ft.
47. 3 ft.
48. 27 : 64.
49. $\frac{64}{27}$ of 270, or 640 cub. ft.
50. $\frac{10}{8}$ of 10 ft., or $12\frac{1}{2}$ ft.

CHAPTER XIX.

1. 40, 25, 35.
2. 100, 200, 300, 400.
3. $3\frac{49}{217}$, $180\frac{140}{217}$, $166\frac{28}{217}$.
4. A has £100, B £200, C £300.
5. £175, £200, and £225.
6. C has 3 times as much, or £3000.
7. Since C had $\frac{5}{27}$, and A's share : B's share = 12 : 10, and also since $1 - \frac{5}{27} = \frac{22}{27}$;

$$\therefore \tfrac{22}{27} : \tfrac{5}{7} = 22 : 5;$$

$\therefore$ C's share was in the business 5 months.

8. 10 months.
9. A £200, and the others £240 each.
10. 100 days.
11. £200.
12. £182, 7s. 2d., etc.
13. 1609.
14. 9 hours. This is nothing but to resolve 24 hours into 2 parts, which are to one another as 8·5″ : 5·1″.
15. 50 pence or 20 piastres.
16. 1078 piastres are really worth £11, 4s. 7d., whereas I only pay 11 sovereigns.
17. 4000.
18. £60.
19. We know that A's profits are to C's profits as 60 : 200 or 3 : 10. We also know that these profits are obtained by A having his money 3 months and C 5 months; hence to

reduce them to the same time, A's profits would be to C's profits as $\frac{8}{3} : \frac{10}{5}$, or 1 : 2. Hence we have to find a ratio = 1 : 2 whose terms differ by 200, which is 200 : 400. Now we know nearly all the elements, and can easily prove that B has his money in 4 months and D 6 months.

20. A invested £500 and B £300.
21. 36 20-mark pieces, and $141\frac{1}{8}$ pfennings.
22. 67 taels 5 mace.
23. 500.
24. 66·25.
25. $\frac{1000}{10}$d.; or 8s. 4d.
26. 480.
27. 480 Swedish kronor, and 530 marks.
28. 49 of each.
29. £4, 3s. 4d., or \$20·00.
30. 290 lepta.
31. $4\frac{1}{6}$, $3\frac{41}{55}$.
32. 110 geese.
33. 1 ox is worth as much as $\frac{11}{4}$ pigs, or 4 oxen = in value 11 pigs.
34. 89 geese.
35. 14 fowls = 3 pigs.
36. 44.
37. 39.
38. 3.
39. $\frac{7}{13}$d.
40. $5\frac{13}{21}$ of the unit of price.
41. Second parcel £16, 10s. 9d., third £14, 3s. 6d.
42. See 41.
43. It loses $123 \times 1\frac{1}{2}$ seconds, and gains $120 \times 1\frac{1}{4}$ seconds, and difference between these divided by 243 will give the average loss.
44. £1, 6s. 8d.
45. £3.
46. 13, 15, 17.
47. £160.
48. £2, 10s.
49. £360.
50. 11s. $1\frac{1}{3}$d.

CHAPTER XX.

1. The share in the public debt nominally worth £100 is being sold for £101, 10s.
2. £20 × 28, or £560.
3. 190.
4. The unfunded debt is a temporary one to be paid off in some definite time; the funded debt has no such limit to its period.
5. The price of stocks and shares increase in value as the dividend becomes due, and are sometimes sold close to the period of the dividends being paid, subject to the condition that the seller receives the dividend when due.
6. £387, 3s. $7\frac{1}{2}$d.
7. 93.
8. $133\frac{1}{3}$.
9. $4\frac{1}{2}$.
10. An increase of £25.
11. 120.
12. £9776.
13. $3\frac{1}{4}$.
14. £5401, 4s. 1d.
15. This is just the same as if there were no brokerage, and the price $63\frac{5}{8}$. Ans. £5411, 8s. $4\frac{3}{4}$d.
16. $99\frac{9}{29}$.
17. £$26,666\frac{2}{3}$.
18. 90.
19. £6120.
20. £127, 10s.
21. £1950.
22. £$4\frac{1}{6}$. For every £88 he receives for $4\frac{1}{2}$ months £$1\frac{1}{2}$, but loses £$\frac{1}{8}$ when he sells out; hence the question to solve really becomes: If I receive £$1\frac{1}{2} - \frac{1}{8}$ for the use of £88 for $4\frac{1}{2}$ months, what do I receive for use of £100 for 12 months?
23. He made £1375, of which he had to give his broker £250.
24. £2890, 10s.
25. 98.

26. The United States 5 per cents. £1, 19s. $4\frac{21352}{35209}$d.
27. $\frac{1}{21}$, $\frac{1}{24}$, $\frac{1}{30}$, or 40, 35, 28.
28. $97\frac{3}{4}$, £126.
29. £3519.
30. £576 and £680.
31. The nominal; the real value is £192 × 85.
32. £8700.
33. 4d.
34. Since for half of the money invested he receives $\frac{3}{90}$ or $\frac{1}{30}$, and for the other half $\frac{5}{110}$ or $\frac{1}{22}$, we must divide the given income into two parts in the ratio of $\frac{1}{30} : \frac{1}{22}$, which gives us £2912, 5s. as the income derived from the 3 per cents., at 90, and this is derived from £87,367, 10s. Hence the money invested is £174,735.
35. £529,200.
36. £14,520.

37. The price will be $\frac{232}{235}$ of $\dfrac{90\frac{3}{4}}{3}$ less $\frac{1}{8}$, or $89\frac{139}{235} - \frac{1}{8}$, or $89\frac{877}{1880}$.
38. £3104.
39. The amount of the principal is not involved, as only percentage is asked. £$108\frac{1}{8}$ earns £2, 10s. interest + £1, 15s. profit less 5d. in the £1, in 3 months. So our proportion is

£$108\frac{1}{8}$: £100 = $\frac{235}{240}$ of $(2\frac{1}{2} + 1\frac{3}{4})$ 4 : what £100 earns,

which gives $15\frac{205}{519}$ per cent.
40. $2\frac{13}{31}$.
41. Subtract from £167 the interest on £110 in the 4 per cents. at par, viz. £$4\frac{2}{5}$, and that on £990 in 5 per cents. at £110, viz. £45, and divide the remainder in the proportion of $\frac{3}{90} : \frac{4}{100} : \frac{5}{110}$, and the interest which the 3 per cents. at 90 give will be found to be £33. Hence there were £990 invested in this stock.
42. 10.
43. 90.
44. £405, 3s. 4d.
45. $93\frac{3}{8}$.
46. £10, 9s. $9\frac{3}{4}$d.
47. 100 : 85, or 20 : 17.

48. £3429.
49. £135 and £139, 14s. respectively.
50. 35 per cent.

Miscellaneous Answers.

I.

1. 15 : 18.
2. 2 : 8.
3. $3\frac{1}{3}$ lbs.
4. $\frac{1}{6}$ if the unit for the price of loaf be pence and that of price of wheat be shillings.
5. $\frac{1}{8}$ penny.
6. 9 and 21.
7. 1004.
8. 19 per 1000. 40000.

II.

9. 14, 21, 28, 35; 30; 150.
10. 44 half feet per quarter second.
11. $\frac{100 \times 36}{60 \times 60}$, or one mile.
12. $13\frac{7}{8}$, $23\frac{1}{8}$.
13. 64.
14. On every 2 oz. you charge $\frac{1}{14}$ of a farthing too little, hence you would be a farthing wrong in 28 oz.
15. 288 times.
16. Any number of inches that will measure 1100 exactly.

III.

17. 317.
18. 13.
19. $4\frac{4}{5}$.

20. $\frac{476}{93}$.
21. $181\frac{1}{7}$, $90\frac{4}{7}$, $45\frac{2}{7}$.
22. 12 : 15 : 20.
23. 5 : 4 : 3.
24. 3000.

IV.

25. 6s. 8d. per £100.
26. £300.
27. $\frac{1}{25}$ penny.
28. Give 4 fourpenny pieces and receive back 1 threepenny piece; or give 3 threepenny pieces and 1 fourpenny.
29. Each man 9s.; woman 8s.; child $4\frac{4}{5}$s.
30. $5\frac{3}{4}$.
31. £40.
32. 2 sevenths.

V.

33. £3, 3s. $10\frac{2}{3}$d.
34. 950.
35. $\frac{36}{60}$.
36. $\frac{12}{21}$.
37. A in $14\frac{2}{17}$ days; B in $10\frac{10}{23}$ days; C in $34\frac{2}{7}$ days.
38. 440.
39. $43\frac{1}{2}$ Italian lire.
40. £720.

VI.

41. 75.
42. 20s. each. Of the 100 sheep, 30 (the 10 on which he loses 10s. and 20 on those of which he gains 5s. each) may be removed. He makes 5×60, or 300s., on the 60 remaining. This 300s. gives him the price of the 10 sheep he loses and $\frac{5}{100}$ of the price of 100 sheep, or the price of 5 sheep. Hence the 300s. pay for 15 sheep, or they are 20s. each.

43. 200.

44. 124365.

45. A and C have to receive 2s. 6d. and 3s. respectively from B.

46. £818, 8s.

47. 12 times and 1 tenth over.

48. 66 ft. a second = 45 miles an hour. Hence, at 10 o'clock the Edinburgh train will be 360 miles from London, after which they approach one another at $(45+45)$ miles an hour, and will meet in $\frac{360}{45+45}$ hours, or at 2 o'clock.

VII.

49. £3, £5, £5.

50. The like figure, say 3, represents 3 tens and 3 nines, or 3 nineteens. Hence, to find the figure, divide 46 by 19, which gives 2 and 8 over. Hence the second figure is 2, and the units $8 \div 2$, or 4.

$$24 \text{ (nonary)} + 24 \text{ (denary)} = 46.$$

51. $3 : 2$.

52. A man earns 5s. a day; woman 3s.; and a child 2s.

53. £625.

54. As in 2 the digit in the third will represent so many twenty-fives and thirty-sixes. Hence, divide the number 89 by 61, which gives a quotient 1 and a remainder 28.

Again, the digit in the second place must represent so many sixes and so many fives, or so many elevens; and this gives 2 and remainder 6, which must be divided by 2 to find digit in units place. Hence answer 123. 123 quinary = 38, and 123 senary = 51, and $38 + 51 = 89$, the number given.

55. $\frac{15}{7}$.

56. $6\frac{8}{11}$, $3\frac{4}{11}$, $26\frac{10}{11}$.

VIII.

57. £2275.

58. £71, 4s. 6d.

59. $\frac{4}{31}$.

60. $17\frac{4}{13}$.
61. 1 cwt. $7\frac{7}{9}$ lbs.
62. 10, 15, 22.
63. 392.
64. $9\frac{3}{13}$d. a score.

IX.

65. If I insure for as many ninety-sevens as there are in £12,000, in case of loss I receive $\frac{12000}{97} \times 100$, or $12000 \times \frac{97+3}{97}$. Now, $12000 \times \frac{97}{97}$ is the value of the property, and $12000 \times \frac{3}{97}$ is the price paid. Hence answer, £$371\frac{13}{97}$.

66. If there is no loss, he is the premium out of pocket. Whereas if his property be lost he saves this as well.

67. 5 more.
68. 124.
69. The 200 women being equal to 160 men, and 160 being the fourth part of 640; for the days to remain equal the ratios must be reduced from 5 to 4, or 20 per cent.

70. See 69.
71. £34, 13s. 4d.
72. $3\frac{3}{5}$ miles an hour.

X.

73. $3\frac{3}{14}$.
74. 5 hours.
75. See 74.
76. £$700 + \frac{585}{97\frac{1}{2}}$ of £$100 =$ £1300.
77. £4.
78. £1100.
79. 1 hour $57\frac{1}{7}$ minutes.
80. 3·3354, etc.

XI.

81. $118\frac{4}{7}$ per cent.

82. $(\frac{1}{2} - \frac{2}{5}) = \frac{102}{100}$ of buying price; $\therefore$ buying price $= \frac{5}{51}$d. His profits on each pair of eggs, viz. $(\frac{1}{2} - \frac{5}{51})$d. $+ (\frac{2}{5} - \frac{5}{51})$d. is contained in £13, 9s. 3d., 4590 times. Ans. 9180 eggs.

83. £540.

84. £30,000 in each.
85. 7 per cent.
86. 10 per cent.
87. 3 : 13.
88. 8 and 5.

XII.

89. 18.
90. £40.
91. The third digit will represent so many twenty-fives, thirty-sixes, and forty-nines; hence to find it, divide the given number by $25+36+49$, or 110. Similarly, to find the second digit divide the remainder 66 by $5+6+7$, or 18; and to find the unit digit divide the second remainder 12 by $1+1+1$, or 3. Ans. 234. 234 (quinary) = 69; 234 (senary) = 94; and 234 (septenary) = 123; and these three numbers added together will give 286.

92. Since the digit in the third place in the septenary is the same as that in the unit place in the other, to find it we must divide the number by $49+1$; but the remainder must be greater than $25+1$, or there would be no part left to be expressed in the other scale; in this case we must only take away from 200, 50×3, and divide this remainder 50 by 26 for the third figure of the one number and the first of the other, and divide this remainder 24 by $5+7$ for the second figure.

Hence, 200 = 321 (septenary) + (123) quinary.

Proof: 321 septenary = 162, and 123 quinary = 38.

93. To find the third place, divide by $25+49$, or 74; this gives a remainder 1, which neither contains $5+7$ or $1+1$ an exact number of times.

94. The greyhound passes over nine of the hare's paces whilst the hare passes over eight. Hence the greyhound gains a pace every eight of the hare's paces. Hence the answer is 1600.

95. The true percentage that a man receives for his money is the interest on the money he could sell his stock for, not on what he gave for it. When the funds are high, money is less valuable.

96. £1 is worth $1+\frac{3}{96}$ at end of first year, and (£$2+\frac{3}{96}$) is worth $(2+\frac{3}{96})\,\frac{99}{96}$ at end of second year, and this is contained in $402\frac{3}{16}$ 192 times. Hence he invested £192.

PART III.

CHAPTER XXI.

1. $200^2 + 2 \times 200 \times 40 + 40^2 + 2 \times 200 \times 7 + 2 \times 40 \times 7 + 7^2$, or 61009.

2. 247.

3. 30.

4. 32.

5. 897.

6. ·1679.

7. 3·478, not 1·1, for the square of 1·1 is 1·21.

8. 2·167.

9. 12·3.

10. $\cdot009 \times \cdot004 = \cdot000036$, and the square root of this is ·006.

11. Reduce $\frac{4}{5}$ to ·800000, etc., and find its root, viz. ·894, or find the square root of 5, and divide 2 by it.

12. If this were asked to five places of decimals, I must reduce it to ten places thus, ·1428571428, which gives ·37796, etc.

13. 80047.

14. 169 (denary) = 2221 (quaternary), which is the square of 31 quaternary and of 13 denary.

15. 1, 4, 13, 24, 41, 100, 121, 144, 213 (notice the three numbers, 100, 121, and 144).

16. 225, 256, 289.

17. $19042^2 - 19041^2$

$$= (19041 + 1)^2 - 19041^2$$
$$= 2 \times 19041 + 1, \text{ or } 38083.$$

18. 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, which all differ by 2.

19. All numbers are even or odd. An even number is divisible by 2; hence its square is divisible by 4.

Take *any* odd number as 43, and write it in the form $42+1$ or $2\times21+1$, and square it as in par. 8, whence we get $4\cdot21^2+2\times2\times21+1$, whence the theorem is evident.

20. Four times.

21. Our statement is this—

$$£1 : £12,278,016 = 2^2 : (\text{weight required})^2.$$

To find this, we shall have to find the square root of 4×12278016, which will be found to be 2×3504, or 7008 carats.

22. 14 and 3.

23. 99 and 74.

24. 128881 denary = 62701 (duodenary), and this is the square of 25*e*.

25. 242, since $243^2-242^2=2\times243+1$.

26. $31^2+27^2=961+729=1690$, and $2\times841+2\times4=1690$.

27. $17956+289=11400\frac{1}{2}+6844\frac{1}{2}$.

28. $58\frac{1}{2}$.

29. $\frac{25}{2}$ ft., or 12 ft. 6 in.

30. 300 yds.

31. A's pace is to B's as 4 : 3.

32. Since $36590401=(6049)^2$, and $12809241=(3579)^2$;

$\therefore$ their diameters are as 6049 : 3579.

33. $2365=43\times55$;

$\therefore$ one number is 49 and the other 6,

and their diameters are as 49 : 6.

But since other pairs of factors besides 43×55, such as 473×5, or 215×11, will produce 2365, we can get other pairs of values, viz. 349 and 344, or 113 and 102.

34. One of the answers is 239 : 234; another, 1183 : 1182; and there are others.

35. Since £1591, 7s. is obtained by multiplying 1500 times by some number, divide £1591, 7s. by 1500, and find the square root of the quotient. The quotient will be 1·0609, which is the square of 1·03; hence 3 is the rate per cent.

36. $2\times1589\times15+15^2$, or 47895.

37. 60.

38. 59·396.

39. If we call the side 1, the diagonal will be the square root of 2, which gives us 1·41421, etc., and this multiplied by 42 will give us, as before, 59·396, etc.

40. If you draw a figure, you will see at once that first you must find the diagonal of one of the sides, as in 39, which gives us 62·22524, etc., and the diagonal of the cube will be the side subtending the right-angled triangle, whose sides are 62·22524 and 44, which would give us 76·2102, etc.

41. The work of 40 is very laborious. Again, let us take the edge of the cube as 1, then the diagonal of one of the faces is $\sqrt{2}$ (square root of 2), and the diagonal of the cube or the side of the right-angled triangle subtending the right angle, whose sides are $\sqrt{2}$, and 1 is $\sqrt{3}$. Find this correct to five or six places, viz. 1·73205, and multiply it by 44, giving the answer 76·2102, etc.

42. We have to find a side where we know the side subtending the right angle, viz. 16, and one of the other sides, viz. $\frac{16}{2}$ or 8; hence the answer is the square root of $16^2 - 8^2$, viz. 13·856, etc.

43. Call the side of the equilateral triangle (to avoid fractions) 2; then the distance from the foot of the perpendicular to one of the angles will be 1, and the length of the perpendicular is $\sqrt{4-1}$ or $\sqrt{3}$, which find, viz. 1·73205, and multiply by 8, which gives 13·8564, as before.

44. Draw a figure, and you will at once see that you have to find a side of the right-angled triangle. Hypotenuse (side opposite right angle) = 50 ft., and one side = 48 ft.; the other is the square root of $50^2 - 48^2$. It is always worth while to test and see whether this has an integral.

Note.—We know $50^2 - 48^2 = (50+48) \times (50-48) = 98 \times 2 = 49 \times 4$, or $7^2 \times 2^2$; hence that part of the street = 14. When we turn the ladder over, we have hypotenuse 50 ft. and one side 40 ft.; hence other side $\sqrt{50^2 - 40^2}$.

Again, $50^2 - 40^2 = (50 + 40)(50 - 40)$, or 90×10, or 9×100, or $3^2 \times 10^2$;

$\therefore$ other part $= 30$, and street is $30 + 14$, or 44 ft. wide.

45. Diagonal $= 291$; hence the saving is $216 + 195 - 291$, or 120 yds.

46. The first triangle will give us the height of the wall within 7 ft. of the top, viz. 56 ft.; therefore the wall is 63 ft. high, and we have to find the third side of the triangle whose sides are 65, 63, which will give us 16 ft.

47. 4.

48. 364 : 2307.

49. 100489, the square of 317.

50. 151 septenary.

CHAPTER XXII.

1. Circles of any diameter, say 5, would contain the unit circle 5×5, or 25 times.

2. A rectangle 3 ft. broad and 5 ft. long contains 15 sq. ft.

3. Rectangles a yard long and a foot broad.

4. If I change the 21 ft. into 7×3 ft., and the 5 ft. 6 in. into 11×6 in., and multiply the 7 by 11, I obtain the number of rectangles 3 ft. long and 6 in. broad there are in the surface, and as each costs 1d. to paint, the answer is 77d., or 6s. 5d.

5. $34\frac{3}{4} \times 26\frac{1}{2} \div 9 \times \frac{9}{2}$ shillings, or £23, 0s. $5\frac{1}{4}$d.

6. $(37\frac{1}{4} + 18\frac{1}{3} + 37\frac{1}{4} + 18\frac{1}{3}) \times 14 \div 2\frac{1}{3} \div 3 \times \frac{1}{2}$ shillings, or £5, 12s. 2d.

7. $14\frac{5}{12} \times 18\frac{7}{12}$ sq. ft., or 29 sq. yds. 6 ft. 131 in.

8. $14\frac{5}{12} \times 18\frac{7}{12} \div \frac{9}{4} \div 3 \times \frac{9}{2}$ shillings, or £8, 18s. $7\frac{5}{18}$d.

9. The best way of finding this is to find the area of the room, viz. 19 ft. 5 by 21 ft. 1 (adding 5 ft. on to each), and subtracting from it the area found in 7, giving an answer of 21 sq. yds. 1 ft.

10. £5, 5s.

11. £12, 19s. $9\frac{3}{16}$d.

12. £27.

13. £2, 15s. $6\frac{1}{4}$d.

14. We can do this as shown in paragraph 7, and in the answer to question 4, viz. reduce the 50 yds. to lengths of

9 in. thus: $\dfrac{50 \times 3 \times 4}{3}$. And if we multiply this by 50, we

obtain the number of rectangles 1 ft. × 9 in., which, if we multiply by $\frac{5}{12}$, we obtain the answer in pounds, viz. £$4166\frac{2}{3}$.

15. £5, 10s. 10d.

16. 29496 ft., or five miles 1032 yds.

17. The $1\frac{1}{2}$ in. thick is of no importance in the working.

Lid and bottom $= 2 \times 3\frac{1}{2} \times 2\frac{1}{2}$ square ft.,

the 4 sides $= (2 \times 3\frac{1}{2} \times 2 \times 2\frac{1}{2}) \times 1\frac{3}{4}$ sq. ft., which added together will be found to contain 38 sq. ft. 72 in.

18. This done in feet, shillings cancel out very prettily. Answer, £8, 18s. 9d., or $\frac{715}{4}$ shillings.

19. 160 yds. £3.

20. First reduce the acres, etc., to sq. yds., and find the square root of the number, viz. 130 yds. Therefore the answer required is 520×195 sq. yds., or 20 ac. 3 rd. 32 po. 2 yds.

21. The internal measure of the sides measured lengthway will be twice $1\frac{1}{2}$ in. less than the external, and $1\frac{1}{2}$ in. less measured downwards, hence the bottom contains 5 ft. × 3 ft. 4 in., and the sides 2×8 ft. 4 in., or altogether $(5 \times \frac{10}{3}) \times (2 \times \frac{25}{3} \times \frac{7}{4})$ sq. feet, or $55\frac{5}{9}$ sq. ft., cost 18s. $6\frac{2}{9}$d.

22. Call the short side 1, then the long side will be $3\frac{1}{2}$, and if we divide 975744 by $1 \times 3\frac{1}{2}$, we obtain the area of our square unit, viz. 278784 sq. ft.; ∴ the unit is 528 ft. each way, hence the field is 528 ft. × 1848 ft., or 176 and 616 yds.

23. 770 yds.

24. Let us call the height 2, then the breadth is 3, and the length 4; the area of its walls contains $2 \times 7 \times 2$, or 28 of these square units; but since 105 shillings contains $\frac{5}{4}$ penny $21 \times 12 \times 4$ times, each of the 28 units contain 36 sq. ft., or a linear unit is 6 ft., hence the height 12 ft., breadth 18, and length 24 ft.

25. As often as $4\frac{1}{2}$ is contained in £2, 12s. 1d. so many square yards are there in the ceiling, viz. $138\frac{8}{9}$, or 1250 sq. ft.; if we take 1 as the breadth and 2 as the length, the ceiling

will contain 2 square units, hence the length is 50 ft. and the breadth 25 ft. In the walls there are as many square yards as £35 contains 2s. 4d., or 300, or 2700 sq. ft., and if we divide this by twice the length × the breadth, or 150, we get the height, 18 ft.

26. 77 yds. 2 ft. 11 in.

27. 908 yds.

28. £117, 15s.

29. This can be done by adopting a unit or by Compound Proportion, and the areas need not be found, £106, 2s. $10\frac{1}{2}$d.

30. 864. £233, 17s. 9d.

31. £176, 17s. $9\frac{9}{25}$d.

32. If the side of a square field is 1, the diagonal is $\sqrt{2}$. If, then, $\sqrt{2}$ represents 90, what does 1 represent? The following proportion will give us it at once:—

$$\sqrt{2} : 1 = 90 : \text{number of feet required};$$

$\therefore$ the length of the side is $\frac{90}{\sqrt{2}}$, and the area of the field $\frac{90 \times 90}{2}$ sq. ft., and this at 1s. for every 9 yds. is

$$£\frac{90 \times 90}{2 \times 9 \times 9 \times 20} = £2, \text{ 10s.}$$

We have worked this out in this full way as an exercise in units, but a figure would show at once that the square on the diagonal of a square is double the square, hence the area of the square is found at once thus: $\frac{90 \times 90}{2}$.

33. 127 yards.

34. 4 minutes.

35. Area to be covered $12\frac{3}{4} \times 8\frac{1}{4}$ sq. ft. $+ 2 \times 21 \times 6\frac{1}{2}$ sq. ft. $= \frac{6051}{16}$ sq. ft., and this at 8 lbs. to the square foot, at $\frac{28}{112}$ shillings a lb., will cost $£\frac{6051 \times 8 \times 28}{16 \times 112 \times 20}$, or £37, 16s. $4\frac{1}{2}$d.

36. £3, 14s. 3d.

37. This problem resolves itself into dividing 49 (the number of square yards) into two parts, so that one multiplied by 16s. 6d., or 33 sixpences, together with the other multiplied by 17 sixpences so as to obtain the sum of £53, 4s. 6d., or 2129 sixpences. This is resolved as shown in Part II., by subtracting 17×49 from 2129 and dividing by 16, which will give us 36 as the number of square yards in the centre, hence the width of the border is $\frac{1}{2}$ of 1 yd., or 1 ft. 6 in.

38. The cost of levelling the original court-yard is to the cost of levelling the large court-yard as 400 : 441, or $20^2 : 21^2$, and the difference between 20 and 21 is 3 feet; hence the answer is 20 yds., and £$\frac{70}{400}$ or 3s. 6d. is price per yard.

39. The edge is 4 ft.; there are therefore 6 squares of 16 sq. ft. each to paint at 6d. a square foot, which would cost £2, 8s.

40. If the figures were drawn we should see that four of the short rows in the former is equal to one of the longer rows in the other, or that the short side in the first is $\frac{1}{4}$ of the long side in the second, or that the short side less 1 ft. is $\frac{1}{4}$ of the longer; in other words, that 4 times the shorter side less the longer is 4 ft., and 12 and 4 will satisfy this, hence 48 is an area in question; but there are others.

41. 30 acres and 70 acres.

42. 45 ft., or 15 yds. at 10s., is 150s.; therefore there are 10 sq. yds., or 90 ft., and this divided by $2\frac{1}{4}$ gives 40 ft., or $13\frac{1}{3}$ yds., at 3s. 6d., or £2, 6s. 8d.

43. 36 sq. ft.

44. 2 ac. 0 rd. 16 po.

45. A square whose side is one-half the side of the original square.

46. The expense of the path is $\frac{12}{19}$ of 513s., or 324s., therefore there are 324 sq. ft. in the path. If, then, we subtract the four corners 4×9, or 36, and divide the remainder $\times 3$, we find the perimeter of the inner square, viz. 96 ft., or each side is 8 yds., which gives us the position of the path.

47. $2\frac{1}{2}$ miles.

48. 16.

49. $\frac{4}{11}$ of an acre.

50. $6 \times 36 \times 36$, or 7776.

CHAPTER XXIII.

1. Spheres.—Spheres vary as the cubes of their radii or diameters; hence a sphere with a diameter of 3 units would contain the unit sphere $3 \times 3 \times 3$, or 27 times.

2. 8 beams, each 2 yds. long and 2 ft. wide and 2 ft. thick.

3. Convert the length into lengths of 9 in., the breadth into lengths of 5 in., and the thickness into lengths of 4 in., and multiply the three numbers together, 90 ft. = 120 nine in., 10 ft. = 24 five in., and 16 in. = 4 four in., hence number of bricks = $120 \times 24 \times 4$, or 11520.

4. $10 \times 10 \times 10 \times 25$, or 25000 unit tetrahedra.

5. 86400.

6. $6\frac{25}{36}$ cub. ft. Care must be taken with the corners. If the two sides of the box be the full 3 ft. 8 in. by 2 ft., the front, back, top, and bottom will be only 6 ft. long.

7. 3229 cub. ft. 1044 in.

8. If you divide $4768\frac{1}{2}$ by 11 ft., you attain the area. Call the width 2 and the length 3, and find the area of this unit, whose side will be found to be $8\frac{1}{2}$, hence the length is $25\frac{1}{2}$ ft. and 17 ft.

9. This fraction gives the answer in pounds. $39\frac{1}{2} \times 3\frac{7}{12} \times 3\frac{7}{12} \times \frac{1}{8}$, or £63, 7s. $1{\cdot}1\frac{35}{48}$d.

10. Do this as in example 3. Ans. 55296.

11. Since 1 ft. $10\frac{1}{2}$ in. contains $4\frac{1}{2}$ exactly 5 times, this also could easily be effected as in 3. Ans. 30,000 bricks.

12. Multiply the number of inches together and find the cube root. Ans. 83 ft. 5 in.

13. 62. $83\frac{1}{3}$ cub. ft.

14. 4 cwts. 1 qr. 24 lbs.

15. 3 tons, 4 cwts. 3 qrs. 4 lbs. 13 oz.

16. £60, 0s. 6d.

17. 2 qrs. 24 lbs.

18. $\dfrac{1}{250000}$ of an inch.

19. 2·2 lbs.

20. 61·162984.

21. A gallon contains 277 cub. in. and weighs $\frac{277}{1728}$ $\times$ 1000 oz.; $\therefore$ a pint weighs $\frac{277 \times 1000}{1728 \times 8}$ oz., and this is equal to 20 $\frac{65}{1728}$ oz., which is 1 lb. and a quarter and a little more than one-third of an oz.

22. 2s. $2\frac{1}{2}$d.

23. This can be done in one effort, reducing to inches, etc., thus—

$$\frac{208 \times 160 \times 1 \times 13 \times 7}{16 \times 2 \times 16 \times 2} \text{ pence, or } £12,\ 6\text{s. } 5\tfrac{1}{2}\text{d.}$$

24. The following fraction gives the cost:—

$$£\frac{9 \times 130 \times 32 \times 13 \times 15}{4 \times 130 \times 20 \times 2 \times 4}, \text{ or } £87,\ 15\text{s.}$$

25. The cubic contents varying as the cubes, and 27 being the cube of 3, the other side must be $12 \div 3$, or 4.

26. 2193.

27. 1·76 ft.

28. 871 ft. 192 in.

29. In the larger room there is a layer of contents containing 1080 cub. ft., of which the height is 3 ft., hence the area of the two rooms is 360 ft., hence the breadth of the larger is 15 ft. and the length of the smaller 20 feet.

30. 70·8, etc., in.

31. Thus simply multiplying together the three numbers (expressed decimally) $94{\cdot}5 \times 1{\cdot}05 \times {\cdot}315$, which gives 31 cub. chains and 255875 cub. links.

32. The weight of a cub. ft. is 12 times $37\frac{1}{2}$ lbs., or 75×6 lbs. The contents of the dish are $\frac{7}{2} \times \frac{7}{2} \times \frac{22}{7} \times \frac{4}{3 \times 12}$ cub. ft., which gives as the weight 1925 lbs., or 17 cwts. 21 lbs.

33. Call the breadth 1, the height 5, and the length 40. Of cubes of this unit there are 200, and therefore each unit contains $91\frac{1}{8}$ cub. ft., and the side of such a cube contains $4\frac{1}{2}$ ft., hence the breadth is $4\frac{1}{2}$ ft.

34. $(4 - \cdot063)^3$ will be found to be $61\cdot023377953$.

35. 244.

36. The area is 100 sq. in., and the heights are 110 ft. and 100 ft., hence cubic content of greater : cubic content of lesser $= 110 \times 100 : 100 \times 100$, or $11 : 10$.

37. 31 cub. yds. 13 cub. ft. $10'\ 3''\ 4'''$, or 1480 in.

38. 32 lbs.

39. 94 cub. ft. $9^{\text{i.}}\ 9^{\text{ii.}}\ 5^{\text{iii.}}\ 6^{\text{iv.}}\ 9^{\text{v.}}$, or $1409\frac{9}{16}$ cub. in.

40. £22, 18s. 4d.

41. 5 miles 1 fur. 35 po. $4\frac{1}{8}$ yds.

42. $33t99e05$.

43. The value of the pearls is a mean proportional between the value of the diamonds and casket. Hence multiply the numbers which represent their value, and find the square root of the product, which will give the price of pearls in shillings, viz. 1197. But since $£3411\cdot9 = 3249$ guineas, the question can be worked in guineas, giving as an answer 57 guineas.

44. The cube root of 729 is 9, hence the diameter of the larger is 9×7, or 63 ft.

45. From the problem we know that £1000 amounts to £1610, 10s. 2·4d., and this amount is obtained from £1000 by multiplying £1000 by some quantity five times in succession.

What, then, we have to do, is to reduce £1610, 10s. 2·4d. to pounds, divide by 1000, and find the fifth root of the quotient

$$£1610,\ 10\text{s.}\ 2\cdot4\text{d.} = £1610\cdot51,$$

and this, divided by 1000, gives $1\cdot61051$. To find the fifth root of $1\cdot61051$—

1	0	00	000	0000	1·61051(1·1
	1	1	1	1	1
	1	1	1	1	
	1	2	3	4	
	2	3	4		
	1	3	6		
	3	6			
	1	4			
	4				
	1				

1	50	1000	10000	50000	61051
	1	51	1051	11051	61051
	51	1051	11051	61051	

Hence the multiplier was 1·1; but as £1 becomes £1·1 at 10 per cent., hence the answer is 10 per cent.

46. £200 amounts to £218·5454 in 3 years, and this, divided by 200, gives 1·092727, and the cube root of this is 1·03. Hence the answer is 3 per cent.

47. 17 per cent.

48. 207360.

49. $\frac{9}{8}$ of £77, 15s. $2\frac{2}{5}$d., or £87, 9s. $7\frac{1}{5}$d.

CHAPTER XXIV.

1. 1 sq. ft. 105·41376, etc., sq. in.

2. $\frac{1}{4}$ of the larger one, viz. 60·4284 sq. in.

3. 40 sq. ft.

4. The altitude of the larger triangle is $2\frac{1}{2}$ ft. more than that of the other.

5. If you draw two triangles of the same shape on the same base line, and through the vertex of the smaller triangle draw a line parallel to the base cutting the sides of the other triangle, and from the points where this line cuts the side you drop perpendiculars on the base, you will find that you have a triangle, whose base is 4 and altitude 3, on the top of a rectangle 4 ft. by the altitude of the smaller triangle. And these two figures we know to contain 30 sq. ft., of which the triangle contains 6, hence the altitude $\times 4 = 24$ ft., or the altitude of smaller triangle $= 6$ ft.

6. 8 sq. ft.

7. 4·2202 sq. chains, or ·42202 ac., or 1 rd. 27·5232 po.

8. The largest rectangle will be half the height of the triangle, and contain half the surface measure of the triangle; $\therefore$ the rest will be $\frac{1}{2} \times 6 \times 6 \times {\cdot}43301$.

9. $6 \times 2^2 \times {\cdot}43301$ sq. chains.

10. $(2 + 2\sqrt{2})\,16$ sq. chains, or $32 + 32 \times 1{\cdot}4141$ sq. chains.

11. 69·57 yds. very nearly.

12. 17 ac. 0 rd. $11\frac{1}{2}$ po.

13. 210 sq. ft.

14. Each side is $\frac{1}{4}$ mile, hence its area $\frac{1}{16}$ square mile or $\frac{1}{16}$ of 640 acres—that is, 40 acres.

15. If it take me 3 minutes to walk one side of a triangular field, 4 minutes to walk the next, and 5 to walk the third, the sides are 3 units, 4 units, and 5 units long, and since $3^2 + 4^2 = 5^2$, therefore the field is a right-angled triangle.

16. Half the square, or 50 sq. in.

17. The altitude of the triangle is double that of the equilateral triangle drawn on the base, which is $5 \times \sqrt{3}$; $\therefore$ the area of triangle is $\frac{1}{2} \cdot 10 \cdot 2 \cdot 5 \cdot \sqrt{3}$ sq. ft., or $50\sqrt{3}$ sq. ft.

18. 40 sq. ft.

19. $10 \cdot \frac{7}{2} \cdot \sqrt{3}$ sq. ft., or $35\sqrt{3}$ sq. ft.

20. Divide the sides of the triangle into three equal parts, and draw through the points lines parallel to the sides, and you will divide the figure into nine equal triangles, each equal to the smaller triangle.

21. Let us call the sides of the triangle and square t and s respectively.

$$\text{Area of triangle} = \frac{\sqrt{3}}{4}t^2,$$

$$\text{and area of square} = s^2.$$

But the areas are equal;

$$\therefore\ t^2 : s^2 = 1 : \frac{\sqrt{3}}{4},$$

$$\text{or } t^2 : s^2 = 4 : \sqrt{3},$$

$$\text{or } t : s = 2 : \sqrt[4]{3}.$$

22. Let us call the sides of the square and hexagon s and h.

$$\text{Area of square} = s^2,$$

$$\text{area of hexagon} = 6 \times \frac{\sqrt{3}}{4}h^2.$$

But the areas are equal;

$$\therefore\ h^2 : s^2 = 1 : 6 \times \frac{\sqrt{3}}{4},$$

$$\text{or } h^2 : s^2 = 4 : 6 \times \sqrt{3},$$

$$\text{or } h : s = 2 : \sqrt{6\sqrt{3}}.$$

23. Lastly, let us call the sides of the hexagon and octagon h and o.

$$\text{Area of hexagon} = 6 \times \frac{\sqrt{3}}{4}h^2,$$

$$\text{and area of octagon} = (2 + 2\sqrt{2})o^2.$$

But these areas are equal;

$$\therefore\ h^2 : o^2 = (2 + 2\sqrt{2}) : 6 \times \frac{\sqrt{3}}{4},$$

$$\text{and } h : o = \sqrt{2 + 2\sqrt{2}} : \frac{\sqrt{6\sqrt{3}}}{2}$$

24. To combine these—

$$t : s = 2 : \sqrt[4]{3}$$
$$= 1 : \frac{\sqrt[4]{3}}{2},$$
$$s : h = \sqrt{6\sqrt{3}} : 2$$
$$= \sqrt{6}\sqrt[4]{3} : 2$$
$$= \frac{\sqrt[4]{3}}{2} : \frac{1}{\sqrt{6}}.$$

I have reduced $\frac{\sqrt{6\sqrt{3}}}{2}$ into $\frac{\sqrt[4]{3}}{2} \div \frac{1}{\sqrt{6}}$.

$$\text{Again, } h : o = \sqrt{(2 + 2\sqrt{2})} : \frac{6\sqrt{3}}{2}$$
$$= 1 : \frac{6\sqrt{3}}{2\sqrt{2+2\sqrt{2}}}$$
$$= \frac{1}{\sqrt{6}} : \frac{6\sqrt{3}}{2\sqrt{6}\sqrt{2+2\sqrt{2}}}.$$

$$\text{Hence } t : s : h : o = 1 : \frac{\sqrt[4]{3}}{2} : \frac{1}{\sqrt{6}} : \frac{3\sqrt{2}}{2\sqrt{2+2\sqrt{2}}}.$$

The reduction of the last quantity may require explanation,

viz. $\frac{6\sqrt{3}}{\sqrt{6}} = 3\sqrt{2}$,

$$\frac{6\sqrt{3}}{\sqrt{6}} = \frac{\sqrt{6}\sqrt{6}\sqrt{3}}{\sqrt{6}} = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}.$$

25. $\frac{1}{2}(24+16)\,6$ or 120 sq. ft.

26. A figure will show at once that the other side is perpendicular to the parallel sides, and therefore 6 ft.

27. A figure would show here also that the other side of the right-angled triangle, of which we know 8 ft. and 6 ft., is $2\sqrt{7}$ ft., hence the other right-angled triangle at the other end of the trapezoid has the two sides about the right angle 6ft. and $(8-2\sqrt{7})$ ft., of which the hypotenuse is $4\sqrt{8-2\sqrt{7}}$ ft.

28. $\sqrt{8 \div \cdot 43301}$ ft. or 4·28 ft.

29. 7.543 sq. ft..
30. 8 in., 16 in., and 18 in.

31. The hexagon contains $54\sqrt{3}$ sq. ft., and the octagon $(2+2\sqrt{2})\ 36$ sq. ft.; and the difference will be found to be 80·2908 sq. ft.
32. 6 ft.
33. 4 in.
34. We can either find the square root of 5, or we can draw a right-angled triangle whose sides containing the right angle are 1 and 2, then the square on the side opposite the right angle will be 5 times as great as the square on the shorter of the other two sides.
35. This is only to find a right-angled triangle, one of whose sides is 3 in., and the area = 3 sq. in., hence the other side is 2 in. long.
36. This is to find a right-angled triangle whose sides are as $1 : 1 : \sqrt{2}$, whose area is $\frac{16}{3}$ sq. in.; hence half a square unit in which the side of the triangle is expressed contains $\frac{16}{3}$ sq. in., and the whole square unit contained $\frac{32}{3}$ sq. in.; hence the side of the triangle is $\frac{8\sqrt{2}}{\sqrt{3}}$, or $\frac{8\sqrt{6}}{3}$, which will be found to be 6·664 in.

37. Through O draw lines parallel to the sides; these will be the four altitudes of the triangles, and since the side of the square is $2\sqrt{3}$ in., the altitude will be $\sqrt{3}$ in., and the

sides of the triangle will be 2 in., and the area of each triangle $\sqrt{3}$ sq. in.; hence the space left will be $4 \times 3 - 4\sqrt{3}$, or $12 - 4\sqrt{3}$ sq. in., or 5·0718 sq. in.

38. 62·5 ac., or 62 ac. 2 rd.

39. £768.

40. From each row $\sqrt{3}$ in. broad I can cut $18+17$ triangles, viz. 18 with their bases on the side of the square and 17 with their vertices on the edge. $\sqrt{3}$ in. is contained in 36 in. $12\sqrt{3}$ times, hence there are 20 rows, or 20×35, that is, 700 triangles which can be cut.

41. The area left is $(36^2 - 700\sqrt{3})$ sq. in., or 183·565 sq. in.

42. As the cardboard from which we have to cut the hexagons is square, it cannot matter whether we cut the first row with their angles or sides on the top side.

Let us have the sides on the top side and the angles on the left side. Between the angles of two hexagons in the same row we must leave 1 inch for the top of the next row of hexagons; hence we can only have 12 hexagons in the first row, and 11 in the second; but these two rows only occupy a surface $3\sqrt{3}$ in breadth, and the next two rows only occupy $2\sqrt{3}$. Let us therefore subtract $3\sqrt{3}$ from 36 for the first two rows and divide the remainder by $2\sqrt{3}$ to see how many more pairs of rows we can have of 12 and 11 respectively, which gives us a quotient of 8·8; we can therefore have 8 pairs of rows and a single one, besides the double row at the top. Hence altogether we can have $(12+11)9+12$, or 219.

43. All the triangles will be equal, hence each is $\frac{1}{8}(2+2\sqrt{2})1$ sq. in., or ·60355 sq. in.

44. The outside dimensions of the frame are 3 ft. $\times$ 2 ft., and the inside $2\frac{2}{5}$ by $1\frac{3}{5}$; hence the area of the frame is

$$6 - \frac{84}{25} \text{ sq. ft., or } \frac{150-84}{25}, \text{ or } \frac{66}{25}, \text{ or } 2\frac{16}{25} \text{ sq. ft.}$$

45. £20.

46. Call the base 1. Altitude of triangle in the hexagon : altitude of triangle in the octagon $= \sqrt{3} : \frac{1}{\sqrt{2}} + \frac{1}{2}$

$$= \sqrt{3} : \frac{2+\sqrt{2}}{2\sqrt{2}}$$

$$= 2\sqrt{6} : 2+\sqrt{2},$$

and the base being the same, these numbers would represent the comparative areas.

47. $\frac{1}{2} \times 12 \times 18$, or 108 sq. in.

48. The length of the string can easily be shown to be $15+15+18$, or 48; if, then, this be increased by 20 in., since the 18 will remain unchanged, the two parts of the string between the nail and the picture are now 25 ft.; hence the distance from nail to picture is $\sqrt{25^2-9^2}$, or $\sqrt{544}$, and the area of wall is $\frac{1}{2} \times 9 \times \sqrt{544} = \frac{1}{2} \times 9 \times 2\sqrt{136} = 9 \times 2 \times \sqrt{39}$ sq. in.

49. Here we have to find the area of a triangle, 18.17.13 (since we know the string over the nail is 30 in.) $= \sqrt{24.6.7.11}$ $= 12\sqrt{77}$ sq. in.

50. Since if the sides were 1 ft. the area would be $6 \cdot \frac{\sqrt{3}}{4} \cdot 1$ sq. ft., as often as this is contained in $18816\sqrt{3}$ sq. ft. so many times squared is 1 ft. contained in the side of the hexagon. Ans. $\sqrt{12544}$, or 112, and this is 84s. or £4, 4s.

CHAPTER XXV.

1. Two quantities though of the same kind which cannot be referred to the same unit, however small, as the diameter and circumference of the same circle.

2. Yes; because $3^2+4^2=5^2$.

3. Approximating to the ratio between the circumference of any circle and its diameter.

4. Draw a regular hexagon in a circle, and you will see that the side is equal to the radius of the circle;—and the perimeter of the hexagon is evidently less than that of the circle, or π is less than 3; but if you draw a regular octagon, whose sides are equal to the radius of the circle, the octagon is entirely without the circle, and therefore its perimeter is greater than the circle, or π is greater than 4.

5. 26 sq. ft. 96 in.

6. $$\text{Area of sector} = \frac{\text{ang.}}{4 \text{ rt. angles}} \times \text{area of circle}$$

$$= \frac{\text{ang.}}{4 \text{ rt. angles}} \cdot \frac{\text{circumference}}{\text{diameter}} (\text{radius})^2$$

$$= \left(\frac{\text{ang.} \times \text{circ.}}{4 \text{ rt. angles}}\right) \frac{\text{radius}}{2}$$

$$= \frac{\text{arc} \times \text{rad.}}{2};$$

$$\text{since } \frac{\text{arc}}{\text{circumference}} = \frac{\text{angle}}{4 \text{ rt. angles}}.$$

7. $\text{Radius} = \frac{100}{2} \times \frac{1}{\pi} = 50 \times {\cdot}31831 \text{ yds.} = 15{\cdot}9155 \text{ yds.}$

8. $$\frac{\text{angle}}{4 \text{ rt. angles}} = \frac{\text{arc}}{\text{circumference}};$$

$$\therefore \text{angle} = \frac{360^\circ{\cdot}8}{2.\pi.10} = 36^\circ \times 4 \times {\cdot}31831 = 45{\cdot}83664^\circ.$$

9. 7·168...ft.
10. ·21... ft.
11. 9 sq. ft.
12. $57^\circ\ 20'$.
13. $\frac{1}{8}\times\pi\times 10000$ sq. miles, or 3926·9875 sq. miles.
14. 4·57... sq. ft.
15. 4 sq. ft.
16. 271·825... sq. ft.
17. 62·8318... sq. ft.

18. $$\left(\frac{100}{2\pi}+4\right)^2\pi-\left(\frac{100}{2\pi}+2\right)^2\pi : \left(\frac{100}{2\pi}+2\right)^2\pi-\left(\frac{100}{2\pi}\right)^2\pi,$$

or 237·699... sq. ft. : 212·566... sq. ft.

19. Area of larger ring is $\pi\times 19$ sq. ft.;
$\therefore$ area of smaller ring is $\frac{3}{10}\times\pi\times 19$ sq. ft.

20. Euc. I. 47 will easily show that the distance of the nearer centre to the common chord is $\frac{3}{2}$ ft., whence the semichord is $\frac{3}{2}\sqrt{15}$ ft.

21. 16·32 sq. ft.
22. The distance between the centres and the radii must be such that any two of them are together greater than the third.
23. 6 ft.
24. 3·14159 : 2·5981.
25. 19·5656 sq. ft.
26. 1 : 2.
27. Four times the semicircle ($\frac{1}{2}.\pi.9$ sq. ft.) contains the square (36 sq. ft.) and the star; $\therefore$ the star $=(2.\pi.9-36)$ sq. ft. $=$ 20·54862 sq. ft.
28. 3·6 yds.
29. 28·005 ft.
30. 4·6685 ft.
31. The faster runner must have run round three times; therefore the course is $\frac{10}{3}$ miles round, and the diameter $=$ $\frac{10}{3}\times$ ·31831, or 1·061... miles.

32. The circumference is 40 ft.;

$\therefore$ radius $= 20 \times \cdot 31831$ ft.;

$\therefore$ area $10 \times \cdot 31831 \times 5 = 15 \cdot 9155$ sq. ft.

33. Perimeter of square : perimeter of circle

$4\sqrt{3 \times 4840}$ yds. : $2\pi\sqrt{3 \times 4840 \times \cdot 31831}$ yds.,

or $4 : 2\pi\sqrt{\cdot 31831}$, or $2 : \sqrt{\pi}$.

34. The breadth is very nearly $28\frac{1}{3}$ rods, and the length 17 rods, hence the expense is £45, 8s. 4d.

35. $\frac{1}{2} \times 15 \times 10\sqrt{3}$ sq. ft., or $75 \times 1 \cdot 7320$ sq. ft.

36. $\pi . \frac{6}{4}$, or $3 \times 1 \cdot 57079$ sq. ft.

37. $\cdot 43301 \times 32$ sq. ft.

38. Area of triangle : area of circle $= \sqrt{21 \times 8 \times 7 \times 6} : 21^2 \times \cdot 31831$, or $4 : 6 \cdot 68451$.

39. 594 entire squares.

40. $17 \cdot 6 \times \cdot 31831$ times.

41. 72 sq. ft.

42. $\pi \times 25$ sq. ft.

43. $51\frac{1}{5}$ sq. ft. See example 49.

44. 18 sq. ft.

45. $45 \cdot 57$ ft.

46. $28 \cdot 28$ miles to the inch.

47. The area of the circumscribed circle is 800 sq. yds.; $\therefore$ radius $\frac{1}{2}(10\sqrt{8})$ or $10\sqrt{2}$, or $14 \cdot 1421$ ft.

48. $9 \cdot 06$ sq. ft.

49. Draw the figure and you will see that the rectangle contains 8 right-angled triangles, whose hypotenuse (100 yds.) is represented by $\sqrt{5}$; and the other sides by 1 and 2, or $\frac{100}{\sqrt{5}}$ yds. and $2 \times \frac{100}{\sqrt{5}}$ yds., and the sides of the rectangle are double these lengths. Hence the area $= 2 \times \frac{100}{\sqrt{5}} \times 4 \times \frac{\cdot 100}{\sqrt{5}}$, or 16000 sq. yds.

50. Area of quadrant : area of semicircle as $\left(\frac{100 \times 2}{\pi + 4}\right)^2 : \left(\frac{100}{\pi + 2}\right)^2$, or $\left(\frac{2}{\pi + 4}\right)^2 : \left(\frac{1}{\pi + 2}\right)^2$.

CHAPTER XXVI.

1. $\frac{1}{3}$ of 20, or $6\frac{2}{3}$ cub. ft.
2. $\sqrt{34}$ ft.
3. $\frac{1}{3} \times \left(\frac{16}{\sqrt{2}}\right)^2 \times 6$, or 256 cub. ft.
4. $4\left(10 \times \frac{16}{\sqrt{2}}\right) + \left(\frac{16}{\sqrt{2}}\right)^2$, or 580·5472 sq. ft.
5. $\frac{1}{2} \times 4 \times \pi \times \sqrt{40}$, or $4 \times \pi \times \sqrt{10}$ sq. ft.
6. $\frac{1}{3} \times \pi \times 4 \times 6$, or $8 \times \pi$ cub. ft.
7. $2 \times 8 \times 2 \times 8 \times \pi$, or 256 sq. ft.
8. $\frac{16 \times 16 \times 16 \times \pi}{6}$, or $\frac{1}{3}$ of 2048 cub. ft.
9. $8^2 \times 16 \times \pi$, or 1024 cub. ft.
10. Since the answer in 8 is $\frac{2}{3}$ of the answer in 9, we can obtain the solid contents of a sphere by finding $\frac{2}{3}$ of the surrounding cylinder.
11. $x = \sqrt{1210 \times 9 \times 144 \times {}^{\cdot}31831}$ in., or $706\frac{1}{2}$ in.
12. $\frac{2^3 \times 1413^3 \times \pi}{1728 \times 27 \times 6}$ cub. yds., or 80622 cub. yds. 20 ft. 1404 in.
13. 46·7658... cub. ft.
14. $10\frac{2}{3}$ cub. ft.
15. $(24000)^3 \times \frac{1}{6\pi^2}$ cub. miles.
16. 14 tons 12 cwts. 0 qrs. 20 lbs. 14 oz.
17. $\left(\frac{7}{4}\right)^3 \times \frac{\pi}{6} \times \frac{1820}{1728}$ oz.
18. $\left(8 - 8 \times \frac{\pi}{6}\right)$, or $8\left(\frac{6-\pi}{6}\right)$ cub. in.

19. 24×4, or 96 cub. in.
20. 76 sq. ft.
21. See 20.

22. $36 \times 4\sqrt{3}$ cub. ft.

23. $\dfrac{10 \times 1760 \times 3 \times 12 \times 2 \times 9 \times 1000}{1728 \times 16 \times 112 \times 20}$ tons.

24. 11·21996 tons.

25. $\dfrac{29 \times 13 \times 1000}{1728 \times 16}$ lbs.

26. $\dfrac{29 \times 13 \times 1000 \times 5^2 \times \pi}{1728 \times 16 \times 112 \times 20}$ tons.

27. $\dfrac{50 \times 12 \times 2 \times 1000}{1728 \times 16}$ lbs.

28. $7 \times \pi + \frac{1}{2} \times \dfrac{2^3 \times \pi}{6}$, or $(7 + \frac{2}{3})\pi$.

29. $2\sqrt{6}$ ft.

30. $(10 - 2 \times {}^{\cdot}31831)2 + \dfrac{8}{6\pi^2}$ cub. ft.

31. $(10 - 2 \times {}^{\cdot}31831)2 + \left(2 \times \dfrac{2}{\pi}\right)$ sq. ft.

32. $30 \times {}^{\cdot}31831$ times.
33. Area of cube : area of sphere $= 6 : \sqrt[3]{36 \times 3{\cdot}14159}$.
34. 58·9032 cub. in.
35. $\sqrt[3]{{}^{\cdot}458379}$ in., or ·77... in.

36. Since the surface = diam. × circ. = diam. × diam. × π = diam.$^2 \times \pi$; ∴ $\frac{1}{3}$surface × rad. = $\frac{1}{3}$diam.$^2 \times \pi \times$ rad. = $\frac{1}{3}$diam.$^2 \times \pi$ $\times \dfrac{\text{diam.}}{2} = \dfrac{\text{diam.}^3 \times \pi}{6}$.

37. $(\frac{1}{3} \times 2 \times \pi \times 4 \times 10 - \frac{1}{3} \times 2 \times \pi \times (\frac{8}{5})^2 \times 4)$ cub. ft., or 62·35 *cub. ft.*

38. The area of the frustum is $\pi\left(8\sqrt{21}-\frac{64\sqrt{19}}{25}\right)$ sq. ft.;

$\therefore$ the silver put on is $\pi\left(8\sqrt{21}-\frac{64\sqrt{19}}{25}\right)\times 144\times\frac{1}{4}\times\frac{1}{1728}$

cub. ft., and this weighs $\pi\left(8\sqrt{21}-\frac{64\sqrt{19}}{25}\right)\frac{1}{48}\times 10500$ oz.,

and this at 5s. an oz. is £$\pi\left(8\sqrt{21}-\frac{64\sqrt{19}}{25}\right)\frac{1}{48}\times\frac{10500}{4}$.

39. $24-1\frac{1}{2}$, or $22\frac{1}{2}$ cub. ft.

40. In arithmetic, of course, this can only be done by finding the solid contents of two cones, whose ratios between their heights and the radii of their bases is equal, *e.g.* compare the solid contents of cones—heights 3 ft. and 5 ft. with radii 6 in. and 10 in. respectively.

Content of smaller : content of larger $=\pi.\times 36\times 3:\pi\times 100\times 5=3^3:5^3$.

41. A figure will make this plain,—the length of the slant side is $\sqrt{109}$; $\therefore$ the extra height will be $\frac{\sqrt{109}}{3}$ of 1 in.

42. The new radius is $\left(36+\frac{\sqrt{109}}{10}\right)$ in. and the new height $\left(120+\frac{\sqrt{109}}{3}\right)$ in.; $\therefore$ the solid contents of the entire cone is

$\frac{1}{3}.\pi.\left(36+\frac{\sqrt{109}}{10}\right)^2\left(120+\frac{\sqrt{109}}{3}\right)$ cub. in.; and if we evaluate this and subtract from it the original cone, viz. $\pi\times 9\times 10\times 1728$ cub. in., we shall get the content of the iron casing.

43. 10301 cub. po.

44. The area of frustum is 29400 sq. ft. Hence the marble required is $\frac{29400\times 12}{1728\times 27}$ cub. yds.

45. There are $20.\pi$ cub. ft., and this weighs $-\frac{20\times\pi\times 20\times 1000}{16}$ oz., and will therefore make $\frac{20\times\pi\times 20\times 1000\times 50}{16}$ coins $=625000\times\pi$ coins.

46. $1728\div\frac{\pi}{6}$, or $6\times 1728\times{\cdot}31831$ spheres.

47. Area of sphere is $2\times 2\times\pi$ sq. in. 1 cub. ft. of gold weighs $\frac{{\cdot}20000}{16}$ lbs.; $\therefore$ $\frac{1}{8}$ lb. (the 10 coins) contains $\frac{16\times 1728}{100000}$ cub. in.; $\therefore$ the thickness is $\frac{16\times 1728}{100000\times 2\times 2\times\pi}$ in.

48. The volume of the gold ball is $\frac{4^3\pi}{6}$; $\therefore$ the volume of the tin ball is $\frac{4^3.\pi}{6}\times\frac{19{\cdot}4}{7{\cdot}3}$, and its area $=\pi\left\{\sqrt[3]{\frac{4^3\times 19{\cdot}4\times 6}{6\times 7{\cdot}3}}\right\}^2$

49. By weighing the iron cylinder, and comparing its weight with what it ought to be, supposing there was nothing in it. The excess would tell you how much of the heavier metal was there. If the excess were two grains, you would have to find a ratio whose terms differed by 2, equal to $\frac{20}{7{\cdot}8}$ or $\frac{100}{39}$, and this would give you the weight of the gold in grains.

50. ${\cdot}0112\ldots$ in. diameter before removing the outside metal.

CHAPTER XXVII.

1. The par of exchange is the relation between the intrinsic value of the coins of two countries; whereas the course of change is the constantly changing relation between their value.

2. On the receipt of the goods, he goes to a banker and buys a bill for the amount, payable in Paris, at the course of

exchange for the day. He sends this to his creditor, who presents it to the person on whom it is drawn in Paris, who pays him the amount.

3. Because coined silver is of less value than the intrinsic value of silver.

4. Pure gold is said to be 24 carats fine; therefore 22 carats fine is $\frac{22}{24}$ of absolute purity. $\frac{22}{24}-\frac{37}{47}=\frac{1034-888}{24\times 47}$, or the former is $\frac{46}{24\times 47}$ degrees finer than the latter.

5. £1 = 25 f. $42\frac{1}{2}$ c.
6. £2346, 10s.
7. 12126 fl. 40 kr.
8. 2425 marks.
9. 425s.
10. 77·401 oz.
11. 5s. $\frac{5}{8}$d. per oz. See method in paragraph 13.
12. See 11.
13. 25 f. $60\frac{7}{9}$ very nearly.
14. See 13.
15. The value of a dollar being $40\frac{59}{109}$d., the answer is £1054, 16s. 4d. very nearly.

16. This resolves itself into the following question:—If 18 oz. cost $\frac{18\times 13}{12}$d., what will 40 oz. cost? Ans. 3s. $7\frac{1}{3}$d.

17. Sov. = 10 fl.; half-sov. = 5 fl.; crown = 2 fl. 50 c.; half cr. = 1 fl. 10 c.; 1s. = 50 c.; 6d. = 25 c.; 3d. = $12\frac{1}{2}$ c.; 4d. = $6\frac{1}{3}$ c.; 1d. = $4\frac{1}{6}$ c.; $\frac{1}{2}$d. = $2\frac{1}{12}$ c.; and $\frac{1}{4}$d. = $1\frac{1}{24}$ c.

18. The large coins, calculated from the sov., will be 1 cr. = 25 c.; half-cr. = 12 c. 5 m.; 1s. = 5 c.; 6d. = 2 c. 5 m.; 3d. = 1 c. $2\frac{1}{2}$ m.; or if calculated from the farthing = 1 c. 2 m.

19. £75, 2 fl. 1 c. $\frac{1}{2}$ m.

20. £10, 11s.

21. That the weight of any volume of the body will be $7\frac{1}{2}$ times as great as that of the equal volume of water.

22. 8.

23. 8 : 4, or 2 : 1.

24. 180 : 189 : 196.

25. 20 : 6 : 15 : 42.

26. In mercury, whose specific gravity is greater than that of iron.

27. As water got colder and colder, it would sink, until the rivers became solid.

28. ·25.

29. 1.

30. $\frac{7{\cdot}8 \times 1000}{1728}$ oz.

31. That it contained some volume of matter within it whose specific gravity was greater than that of iron.

32. 3·6.

33. 8000 oz.

34. In salt, because the salt water being heavier than the fresh water you displace, it presses upwards with more power than the fresh does.

35. 7791·304.

36. To the buyer, the heavier the air the less would they press down the spring.

37. Slightly less, or he couldn't float.

38. No; it varies according to the temperature.

39. ·79. Let W_w, W_a, represent the weight of piece of copper in water and air respectively, and let S_c and 1 represent the gr. of copper and water: then $\frac{W_c}{W_a - W_w} = \frac{S_c}{1}$;

$\therefore \frac{W_w - W_a + W_w}{W_a} = \frac{S_c - 1}{S_c}$ (see ch. xi. par. 10); $\therefore \frac{W_w}{W_a} = \frac{S_c - 1}{S_c}$;

$\therefore W_a = \frac{W_w\, S_c}{S_c - 1}$; hence the operation becomes one of pure arithmetic.

40. 5·06 ft.
41. See 40.
42. 1520 oz.
43. 5 : 8.
44. Twice as much fresh water as there was sea water.
46. 3 : 2.
47. 13 oz.
48. 16 oz.
49. 275 oz.
50. This proportion would give it—

$$\begin{matrix} 19{\cdot}4 : {\cdot}24 \\ 1 \text{ cub. in.} : 1 \text{ cub. yd.} \end{matrix} = 224\tfrac{1}{2} \text{ dwts. : weight required,}$$

$$\text{or } \frac{24 \times 27 \times 1728 \times 449 \times 24}{1940 \times 1 \times 2 \times 7000} \text{ lbs. avoir.}$$

Of course we multiply by 24 to reduce the dwts. to grains, and divide by 7000 to reduce them to lbs.

MISCELLANEOUS ANSWERS.

1. $4^3 = (2 \times 2) \times (2 \times 2) \times (2 \times 2)$
$= (2 \times 2 \times 2) \times (2 \times 2 \times 2) = 8 \times 8 = 8^2.$
2. 607.
3. 35611289.
4. 73, 75, 77, 150.
5. $5 + 6 + 6 + 0 + 4 = 21 = 3 \times 7.$
6. 908.
7. 54.
8. 13s. 4d.
9. £326, 15s. $11\frac{19}{51}$d.
10. $3333\frac{1}{3}$.
11. 84, 147, 392.

12. 6, 4, 3, 2.
13. 1854.
14. 999.
15. 12143; 151152442.
16. See 15.
17. 5 per cent.
18. Discount = £5. Principal = £505.

19. Since the interest is obtained by multiplying the principal by the rate and the time, and dividing the result by 100, the interest taken 100 times is equal to the principal multiplied by the rate and the time; hence the rate can be found as in the formula.

20. He is only receiving $\frac{3}{102}$ instead of $\frac{3}{92}$; $\therefore$ he loses on each £100, $\pounds\frac{3(51-46)100}{2\times 51\times 46}$, or $\pounds\frac{5\times 25}{17\times 23}$.

21. 15 ft. 4 in.
22. 192 cub. ft. 2304 sq. ft.
23. £315.

24. The more men there are the less time it will take to do a piece of work.

25. $\frac{13}{60}$.
26. Because $1\frac{1}{2}$d. will divide by 3.
27. $\pounds 4\dfrac{7\frac{6\frac{1}{2}}{12}}{20}$.
28. £3, 8s. $4\frac{1}{2}$d.
29. £2, 4s. 8d.
30. $4\frac{8}{33}$ lbs.
31. $322\frac{2}{3}$ quarter ft.
32. $2\frac{14}{15}$.

33. I lose £2 on that which is nominally worth £100, but for which I may have given more or less than £100, according to the price of the stock when I bought it.

34. I pay $94\frac{5}{8}$, and receive $1\frac{1}{2}+92\frac{7}{8}$, thereby losing $\frac{1}{4}$ per cent. in 3 months.

35. $58\frac{795}{2216}$.
36. 12.
37. £111, 2s. $2\frac{2}{3}$d.

38. If one quantity is a multiple of a second, the second is the reciprocal of the same multiple of the first; *e.g.* if A is three times as old as B, B's age is $\frac{1}{3}$ of A's, and $\frac{1}{3}$ is the reciprocal of 3.

39. 1000 francs.

40. The exchange value of a dollar, nominally worth 4s. 6d., is only $\frac{100}{108\frac{1}{2}}$ of 4s. 6d.

41. 25 f. 61 c.

42. $3\frac{9}{10}$ per mille nearly.

43. 7·776.

44. $232\frac{19}{108}$ oz.

45. $6567\frac{11}{42}$ oz., or 3 cwt. 2 qrs. 18 lbs. $7\frac{11}{42}$ oz.

46. 2 cwt. 1 qr. 3 lbs. $5\frac{5}{7}$ oz.

47. $12\frac{3}{4}$ and $4\frac{1}{4}$ cub. ft. respectively.

48. ·9125.

49. £1, 1s. $10\frac{1}{2}$d. per gallon.

50. Discount is calculated on the future value of some money.

51. A sum of money now is worth some larger sum at some future time, and this larger sum is called the amount of the smaller; and as it is on this larger amount that the discount is calculated, it may be said that the principal of the discount question is the amount of the interest one.

52. $16\frac{1}{14}$ per cent.

53. He paid £$\frac{12}{13} \times 70\left(\frac{4}{3}+\frac{5}{4}+\frac{20}{17}+\frac{10}{9}+\frac{20}{19}\right)$.

As the common denominator of these fractions would be very large, it would be quicker to find out the price of each batch and add them together. To find the above expression is a useful mental exercise.

54. £8.

55. 3·795 nearly.

56. A number between 1 and 2 which, if multiplied by itself, will produce 3. Such a number cannot be exactly found, but we can find one as near to it as we like.

57. By definition $\sqrt{5} \times \sqrt{5} = 5$; or $\sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$.

58. $\sqrt{\frac{3}{5}} = \sqrt{\frac{3 \times 5}{5 \times 5}} = \frac{1}{5}\sqrt{15}$.

59. $\frac{105}{175}$.

60. $\frac{30}{50}$.

61. 15 : 25.

62. 123. **63.** 160 lbs.

64. 195°. **65.** 22·35431 sq. ft.

66. $6\frac{6}{13}$ months before end of year, or on June 19.

67. $4\frac{5}{51}$ mo., note $\dfrac{202 \times 2 + 406 \times 3 + 412 \times 6}{202 + 406 + 412}$.

68. This shows that the present worth of a sum of money for a given time cannot be calculated from a present worth calculated for another time, unless the rate be known.

69. 23 months hence nearly.

70. 6, 50, 256.

71. The first might be any multiple of 3.

72. 193 roubles 14 copecks.

73. ·86602 sq. in.

74. $2 = 2$. $4 \times 8 = 4 \times 7 + 4$. $5 \times 64 = 5 \times 63 + 5$. $3 \times 512 = 3 \times 511 + 3$. Since 4×7, 5×63, and 3×511 will all divide by 7, and since also $2 + 4 + 5 + 3 = 7 \times 2$, the number will divide by 7.

75. 70043957.

76. 37.

77. 20 years.

78. Because the number of years appears as an index, and it requires a knowledge of Logarithms to find an index, except by trial.

79. It must be placed 30 ft. from the fulcrum.

80. 10000, and its square root is 100 or 25 (denary).

81. $\frac{19}{56}$ gallon.

82. £16, 18s. $6\frac{1}{2}$d.

83. £328, 10s. 10d.

84. $7\frac{31}{47}$ days.

85. Proportion is the equality of two ratios; *e.g.* the ratios between 5 men and 3 men, and between £10 and £6 being the same; 5 men : 3 men = £10 : £6 is a proportion.

86. 2s. 3d. a lb.

87. ·896 sq. ft.

88. 40 lbs.

89. The balloon being lighter than air, the pressure of the air from below makes it rise.

90. 5 tons 3 cwt. 3 qrs. 4 lbs. 8 oz.

91. $\frac{10500}{2153}$, or $4\frac{1888}{2153}$ in.

92. See 91.

93. 9 : 4.

94. 23712.

95. £96.

96. 125, 166, 213, 266; which are expressed by 325 in the senary, septenary, octonary, and nonary scales respectively. The next number is 266 + 4, and 266 + any number of fours could be so divided.

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